Why knot? Algebraic coloring invariants of Legendrian knots Mellon Forum Grace Hopper College

#### Lực Ta

Advisor: Sam Raskin

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### Historical background

- Coloring knots
- The Legendrian classification problem
- 2 GL-racks
- 3 Distinguishing results
- 4 Computer search
- 5 Algebraic results: A bird's-eye view

# Let's play a game. . . (1/2)



# Let's play a game...(2/2)



### Reidemeister moves

Two knots are equivalent if and only if they're related by these moves:



### An early way to distinguish knots

A knot is *tricolorable* if we can color its strands in the following way:

- We use exactly 3 colors (no more, no fewer).
- The three strands at each crossing either (a) all share the same color or (b) all have different colors.



### Why is tricolorability a "knot invariant"?



Why knot? Algebraic coloring invariants of Legendrian knots

### The standard contact structure

**Legendrian knots** are knots with a certain restriction on their shape, determined by something called the *standard contact structure*:



- When y = 0, the planes are flat.
- When moving in the positive *y*-direction, the slopes grow more negative.
- When moving in the negative *y*-direction, the slopes grow more positive.

### Legendrian knot diagrams



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- We can use a gadget called a **GL-rack** to color Legendrian knots (hence distinguishing them).
- GL-racks are expansions of *racks* and *quandles*, which are gadgets that color non-Legendrian knots.
- We can assign a GL-rack  $\mathcal{G}(\Lambda)$  to every Legendrian knot  $\Lambda$ .
  - Colorings of  $\Lambda$  must satisfy certain criteria imposed by  $\mathcal{G}(\Lambda).$
- To distinguish Legendrian knots, it suffices to distinguish their GL-racks. And that's when we get to "color"!



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## Solving an old conjecture (1/2)

- Like before, we construct GL-racks  $\mathcal{G}(\Lambda_1)$  and  $\mathcal{G}(\Lambda_2)$ .
- These GL-racks satisfy certain criteria determined by cusps, strands, and crossings.



#### Theorem

 $\Lambda_1$  and  $\Lambda_2$  are distinct.

#### Proof sketch.

- ${\small \textcircled{O}}$  Using a computer search, we look for ways to color  $\Lambda_1$  and  $\Lambda_2$  with GL-racks.
- Interpretent of the second second
  - There are 3 ways to color  $\Lambda_1$  by L under the criteria of  $\mathcal{G}(\Lambda_1)$ .
  - However, there aren't any ways to color  $\Lambda_2$  by L like this!
- § Since their coloring numbers are different,  $\Lambda_1$  and  $\Lambda_2$  are distinct.

### Solving recent conjectures (1/2)

Similarly, we can distinguish these two Legendrian knots...



- ... and these two Legendrian knots.
- This solves several open questions posed in 2023!



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# Enumeration (1/2)



- *Quandles* are a specific type of rack. They were invented 10 years before racks were.
- We can count the algebraic gadgets we've seen today based on how many "colors" they contain:

Number of "colors"	0	1	2	3	4	5	6	7	8
GL-racks	1	1	4	13	62	308	2132	17268	189373
GL-quandles	1	1	2	6	19	74	353	2080	16023
Racks	1	1	2	6	19	74	353	2080	16023
Quandles	1	1	1	3	7	22	73	298	1581

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A brief overview of some even more abstract findings...

- There's a natural one-to-one correspondence between racks and GL-quandles.
  - Cool because racks and GL-quandles were developed independently!
- We can use *symmetries* to...
  - classify several infinite families of GL-racks.
  - better understand how different GL-racks work.
  - understand how GL-racks interact with one another.
- Certain GL-racks allow for other methods of distinguishing Legendrian knots.
  - These methods may be more effective than coloring in certain cases.
  - They also have a natural relationship to algebraic gadgets called *tensor products*.

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## Thank you!



# Generalized Legendrian racks: Classification, tensors, and knot coloring invariants

Luc Ta

Contact: luc.ta@yale.edu | luc-ta.github.io

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Lực Ta