

## COUNTEREXAMPLE ABOUT LIE RACKS

LÙC TA

The purpose of this writeup is to propose infinitely many pairwise nondiffeomorphic counterexamples to the following theorem.

**Theorem 1** ([1, Thm. 4.6]). *Let  $(M, \triangleright, \mathbf{0})$  be a Lie rack integrating a Leibniz algebra  $\mathfrak{g}$ . Then  $\mathfrak{g}$  is a Lie algebra if and only if  $X$  is a quandle. In that case,  $(M, \triangleright, \mathbf{0})$  is isomorphic to the conjugation quandle of a Lie group integrating  $\mathfrak{g}$ .*

For every integer  $n \geq 2$ , we construct a Lie rack  $(M, \triangleright, \mathbf{0})$  that integrates a Lie algebra  $\mathfrak{g} = (T_{\mathbf{0}}M, [\cdot, \cdot])$  despite not being a quandle. The underlying manifold  $M$  is the disjoint union of  $n$  copies of the real line, that is,

$$M := \mathbb{R} \times \{0, 1, \dots, n-1\}$$

equipped with its canonical manifold structure. Pick the basepoint of  $M$  to be  $\mathbf{0} \in M$ , and define a (left-distributive) rack operation  $\triangleright$  on  $M$  by

$$(x, k) \triangleright (y, \ell) := \begin{cases} (y, \ell) & \text{if } k = 0, \\ (-y, \ell) & \text{if } k \geq 1. \end{cases}$$

Checking the following is straightforward.

**Proposition 2.** *The triple  $(M, \triangleright, \mathbf{0})$  is a Lie rack that is not a quandle.*

On the other hand, we also have the following.

**Proposition 3.** *Let  $\mathfrak{g} = (T_{\mathbf{0}}M, [\cdot, \cdot])$  be the (left) Leibniz algebra associated to  $(M, \triangleright, \mathbf{0})$ . Then  $\mathfrak{g}$  is abelian; in particular,  $\mathfrak{g}$  is a Lie algebra.*

*Proof.* We have to show that  $[X, Y] = 0$  for all  $X, Y \in \mathfrak{g}$ . By definition,

$$[X, Y] = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\gamma(t)}(X) = \left. \frac{d}{dt} \right|_{t=0} ((dL_{\gamma(t)})_{\mathbf{0}}(X)),$$

where  $\gamma: [0, 1] \rightarrow M$  is a smooth path such that  $\gamma(0) = \mathbf{0}$  and  $\gamma'(0) = Y$ .

Since  $\gamma$  is continuous, its image is contained in the same connected component of  $M$  as  $\mathbf{0}$ . In other words, for all  $t \in [0, 1]$ , we have  $\gamma(t) \in \mathbb{R} \times \{0\}$ , so  $L_{\gamma(t)} = \text{id}_M$ . Therefore,  $(dL_{\gamma(t)})_{\mathbf{0}} = \text{id}_{\mathfrak{g}}$ , so  $[X, Y] = 0$ .  $\square$

### REFERENCES

- [1] G. La Rosa and M. Mancini, *Two-step nilpotent Leibniz algebras*, Linear Algebra Appl. **637** (2022), 119–137. MR4355961