Generalized Legendrian racks: Classification, tensors, and knot coloring invariants

Math 475: Senior Essay
Final Presentation

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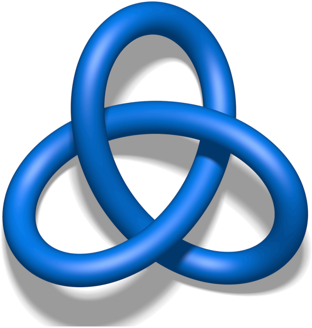
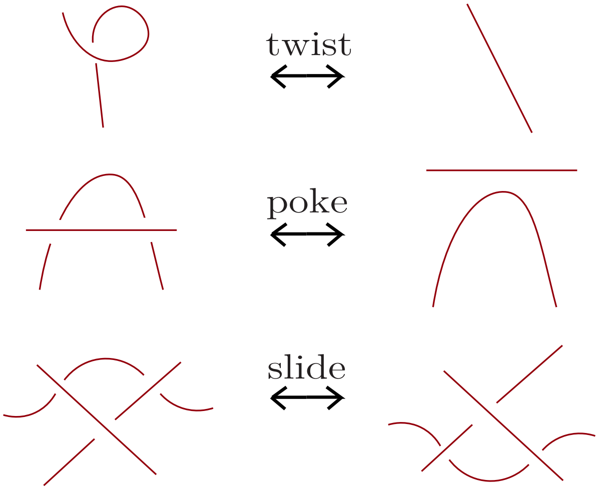
Outline

Motivation: Distinguishing knots



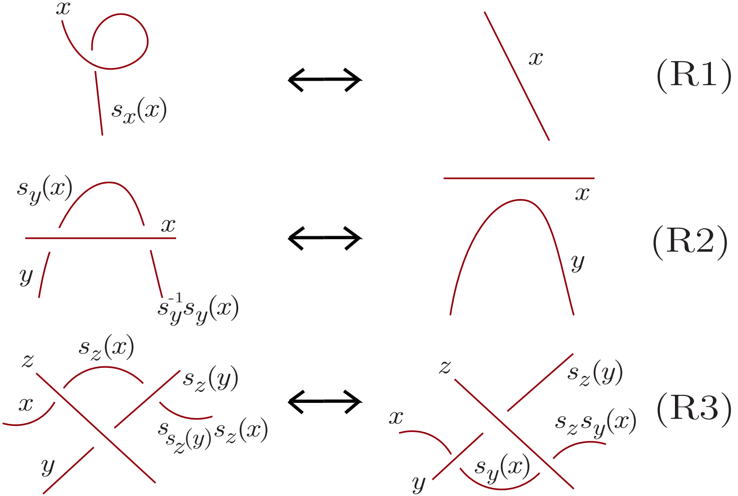
Reidemeister moves (1/2)

Two knots are equivalent up to ambient isotopy if and only if they're related by a finite sequence of *Reidemeister moves*.



Reidemeister moves (2/2)

These moves inspire algebraic objects called *racks* and *quandles*.



Racks and quandles (1/2)

Definition

Let X be a set, let $s : X \to S_X$ be a map, and write $s_x := s(x)$.
We call the pair (X, s) a **rack** if

$$s_x s_y = s_{s_x(y)} s_x$$

for all $x, y \in X$. If in addition $s_x(x) = x$ for all $x \in X$, then we say that (X, s) is a **quandle**.

Example (Permutation racks)

Fix a permutation $\sigma \in S_X$, and define $s_x := \sigma$ for all $x \in X$. Then $(X, \sigma)_{\mathrm{perm}} := (X, s)$ is a rack, and it's a quandle if and only if $\sigma = \mathrm{id}_X$.

Racks and quandles (2/2)

Here are two examples of quandles.

Example (Conjugation quandles)

Let X be a union of conjugacy classes in a group G, and consider the conjugation maps

$$c_x^G(y) := xyx^{-1}.$$

Then $\operatorname{Conj} X := (X, c^G)$ is a quandle.

Example (Takasaki kei)

Let A be an abelian group. For all $a, b \in A$, define

$$s_b(a) := 2b - a.$$

Then $T(A) := (A, s)$ is an involutory quandle.

Rack homomorphisms

Definition

A map $\varphi : (X, s) \to (Yt)$ is a **rack homomorphism** if, for all $x \in X$,

$$\varphi s_x = t_{\varphi(x)} \varphi.$$

Example

Group homomorphisms $\varphi : G \to H$ are also quandle homomorphisms $\varphi : \operatorname{Conj} G \to \operatorname{Conj} H$:

$$\varphi c_x^G(y) = \varphi(xyx^{-1}) = \varphi(x)\varphi(y)\varphi(x)^{-1} = c_{\varphi(x)}^H \varphi(y).$$

Example

Every rack structure $s : (X, s) \to \operatorname{Conj} S_x$ is a homomorphism:

$$ss_x(y) = s_{s_x(y)} = s_{s_x(y)} s_x s_x^{-1} = s_x s_y s_x^{-1} = c_{s_x}^{S_X}(s_y) = c_{s(x)}^{S_X} s(y).$$

Categorical centers

Recall: The **center** of a category $\mathcal{C}$ is the commutative monoid $Z(\mathcal{C})$ of natural endomorphisms of the identity functor $\mathbf{1}_{\mathcal{C}}$.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Y & \xrightarrow{\eta_Y} & Y \end{array}$$

Essentially, elements of $Z(\mathcal{C})$ are collections η of morphisms in $\mathcal{C}$ that commute with all other morphisms φ in $\mathcal{C}$.

Example

Let A-mod be the category of modules over a ring A. Then the categorical center $Z(A\text{-mod})$ is isomorphic to the ring-theoretic center $Z(A)$ of A.

A canonical rack automorphism

Every rack $R = (X, s)$ has a canonical automorphism θ_R defined by

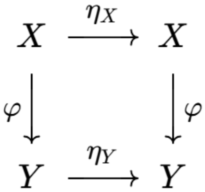
$$x \mapsto s_x(x).$$

Also, R is a quandle if and only if $\theta_R = \mathrm{id}_X$.

Theorem (Szymik, 2018)

Let Rack *be the category of racks. Then the collection of θ_R's generates the categorical center $Z(\mathrm{Rack}) \cong \mathbb{Z}$.*

That is, the following diagram commutes for all rack homomorphisms $\varphi : X \to Y$ if and only if $\eta_X = \theta_X^k$ for some $k \in \mathbb{Z}$, and same for Y:



Racks generalize knot groups

Theorem (Joyce, 1982)

Let Adconj : Rack $\to$ Grp *be the left adjoint to the functor* Conj : Grp $\to$ Rack*. If* $\mathcal{Q}(K)$ *is the "fundamental quandle" of a link* $K \subset S^3$*, then*

$$\mathrm{Adconj}\, \mathcal{Q}(K) \cong \pi_1(S^3 \setminus K).$$

Theorem (Fenn–Rourke, 1992)

A similar result holds for framed links $N(L)$ *in connected 3-manifolds* M*, but with*

$$\pi_2(M, \overline{M \setminus N(L)})$$

instead of $\pi_1(S^3 \setminus K)$*.*

GL-racks

Definition

A **GL-structure** on $R = (X, s)$ is a rack automorphism $\mathtt{u} \in \operatorname{Aut} R$ such that $\mathtt{u}s_x = s_x\mathtt{u}$ for all $x \in X$. We call $(R, \mathtt{u})$ a **GL-rack**.

Example (Permutation GL-racks)

GL-structures on permutation racks $(X, \sigma)_{\mathrm{perm}}$ are permutations $\mathtt{u} \in S_X$ that commute with σ.

Example (Conjugation GL-quandles)

Given a group G and a central element $z \in Z(G)$, multiplication by z is a GL-structure on $\operatorname{Conj} G$.

GL-rack homomorphisms

Definition

A **GL-rack homomorphism** is a rack homomorphism that commutes with/intertwines GL-structures.

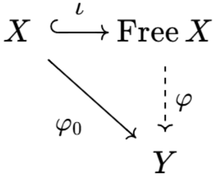
Example

- If $(R, \mathtt{u})$ is a GL-rack, then θ_R and $\mathtt{u}$ are GL-rack automorphisms.
- In particular, we have an automorphism

$$\mathtt{d} := \theta_R^{-1}\mathtt{u}^{-1}.$$

Free racks and GL-racks (1/2)

Due to universal algebra, **free racks** and **free GL-racks** exist and are characterized by a universal property similar to that of free groups or free modules:



Example

- The free rack on one generator is the permutation rack $(\mathbb{Z}, \tau)_{\mathrm{perm}}$, where $\tau : \mathbb{Z} \to \mathbb{Z}$ is the translation $k \mapsto k + 1$.
- The free GL-rack on one generator is the permutation GL-rack $(\mathbb{Z}^2, \sigma)_{\mathrm{perm}}, \mathtt{u}$, where σ and $\mathtt{u}$ increment each coordinate by 1.

Free racks and GL-racks (2/2)

The free rack on a set X is constructed in more or less the same way we construct free groups.

1. Let the *universe of words* $W(X)$ be the set containing X and $x, s_y(x), s_y^{-1}(x)$ for all $x, y \in W(X)$.
2. Let $V(X)$ be $W(X)$ modulo the congruence generated by the relations

$$s_y^{-1} s_y(x) \sim x \sim s_y s_y^{-1}(x) \quad \text{and } s_z s_y(x) \sim s_{s_z(y)} s_z(x).$$

3. This gives us a rack structure $s \in S_{V(X)}$, so we define

$$\mathrm{Free}_{\mathrm{Rack}}\, X := (V(X), s).$$

Free GL-racks are defined similarly.

The history of GL-racks

- The earliest predecessors to GL-racks were introduced by Kulkarni–Prathamesh (2017).
- Ceniceros–Elhamdadi–Nelson (2021) introduced *Legendrian racks*, which are GL-racks in which $\mathtt{u} = \mathtt{d}$; that is,

$$\theta_R = \mathtt{u}^{-2}.$$

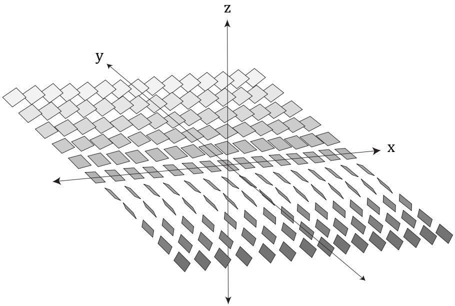
- GL-racks were introduced independently by Kimura (2023) and Karmakar–Saraf–Singh (2023).

Outline

The standard contact structure

Definition

The **standard contact structure** on $\mathbb{R}^3$, denoted by ξ_{std}, is an assignment of a plane to each point (x, y, z) defined by $dz - y\,dx = 0$.

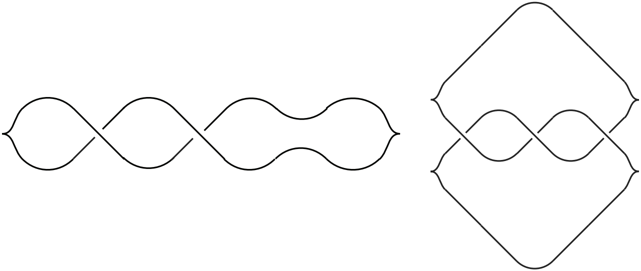


When $y = 0$, the planes are flat. When moving in the positive y-direction, the slopes grow more negative; when moving in the negative y-direction, the slopes grow more positive.

Legendrian knots

Definition

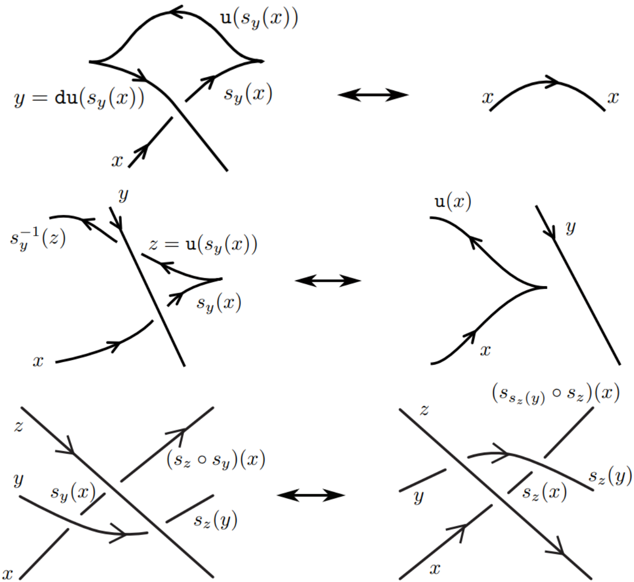
A smooth knot is called **Legendrian** if it lies everywhere tangent to ξ_{std}.



Legendrian knots are usually studied via their *front projections* onto the *xz*-plane, viewed from the negative *y*-axis.

- Cusps instead of vertical tangencies
- Only one type of crossing

Legendrian Reidemeister moves

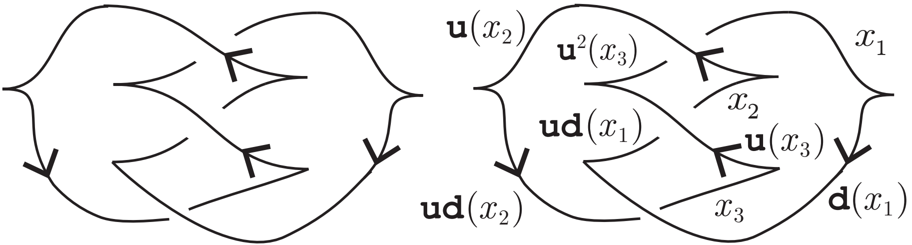


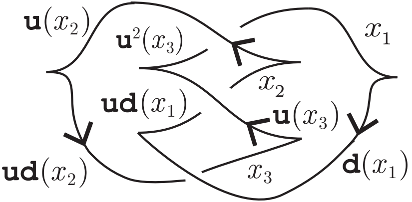
Example of $\mathcal{G}(\Lambda)$: Generators (1/2)

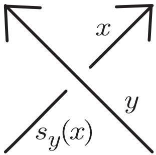
Idea

- To each Legendrian link Λ, we'll assign a GL-rack $\mathcal{G}(\Lambda)$ whose isomorphism type is a Legendrian invariant.
- The idea is to convert Legendrian front projections into quotients of free GL-racks.

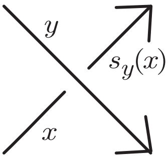
Let's compute the GL-rack of the Legendrian trefoil shown below.







(N)



(P)

$\mathcal{G}(\Lambda)$ is the free GL-rack on three generators x_1, x_2, x_3 modulo the congruence generated by the relations

$$\begin{cases} s_{\mathtt{d}(x_1)}(x_3) = \mathtt{ud}(x_2), \\ s_{\mathtt{u}(x_3)}(x_2) = \mathtt{ud}(x_1), \\ s_{\mathtt{u}(x_2)}(x_1) = \mathtt{u}^2(x_3), \end{cases}$$

which we obtain by applying (N) or (P) at each crossing.

Coloring Legendrian knots

To distinguish Λ from some other Legendrian knot Λ', it suffices to show that $\mathcal{G}(\Lambda)$ and $\mathcal{G}(\Lambda')$ are nonisomorphic.

To do *that*, it suffices to find a GL-rack L such that

$$|\operatorname{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda), L)| \neq |\operatorname{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda'), L)|,$$

an inequality of **coloring numbers**.

Remark

Later, we'll discuss a "nice" subcategory of GL-racks that *enhance* the coloring number as a Legendrian invariant.

Motivations for studying GL-racks

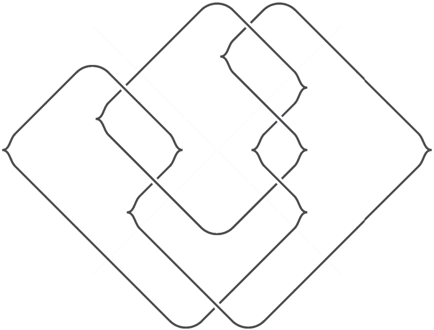
Question (Kimura, 2023)

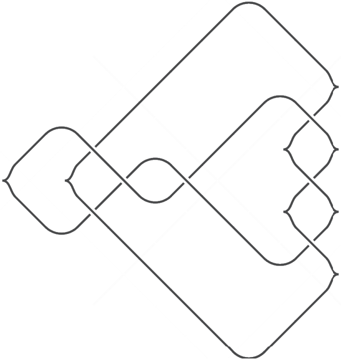
Can GL-racks distinguish Legendrian knots with the same classical invariants?

Answer (T., 2025)

Yes—even better, GL-racks distinguish between Legendrian knots with the same homological invariants!

Filling out the Legendrian knot atlas (1/6)



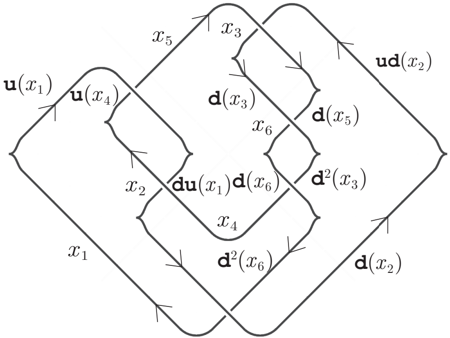


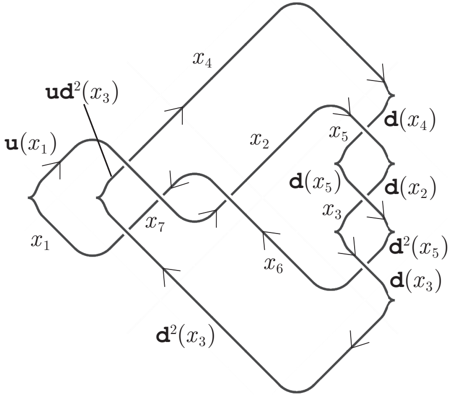
Question (Chongchitmate–Ng, 2013)

Are these Legendrian 6_2 knots distinct?

Answer (Dynnikov–Prasolov, 2021)

Yes! Just give us 146 pages to develop the machinery.





Let Λ_1 and Λ_2 be these Legendrian knots. We'll distinguish their GL-racks using coloring numbers.

Filling out the Legendrian knot atlas (3/6)

We compute $\mathcal{G}(\Lambda_1)$ and $\mathcal{G}(\Lambda_2)$ to be free GL-racks modulo the congruences generated by the relations

$$\mathcal{G}(\Lambda_1)\begin{cases} s_{x_1}\mathtt{u}(x_4) = x_5, & s_{x_4}\mathtt{du}(x_1) = x_2, \\ s_{x_2}(x_1) = \mathtt{d}^2(x_6), & s_{x_5}(x_3) = \mathtt{ud}(x_2), \\ s_{x_3}(x_6) = \mathtt{d}(x_5), & s_{x_6}(x_4) = \mathtt{d}^2(x_3). \end{cases}$$

$$\mathcal{G}(\Lambda_2)\begin{cases} s_{x_1}\mathtt{ud}^2(x_3) = x_4, & s_{x_5}(x_3) = \mathtt{d}(x_2), \\ s_{x_1}(x_6) = x_7, & s_{x_3}(x_6) = \mathtt{d}^2(x_5), \\ s_{x_6}(x_2) = \mathtt{u}(x_1), & s_{x_3}(x_7) = x_1. \\ s_{x_2}(x_5) = \mathtt{d}(x_4), & \end{cases}$$

Filling out the Legendrian knot atlas (4/6)

Theorem

Λ_1 and Λ_2 are Legendrian nonisotopic.

Proof sketch.

Using a computer search, we look for GL-racks L such that

$$|\operatorname{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda_1), L)| \neq |\operatorname{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda_2), L)|.$$

Such an L does exist! Namely, let $\sigma \in S_3$ be the 3-cycle (123). Consider the permutation GL-rack

$$L := ((\{1,2,3\}, \sigma)_{\mathrm{perm}}, \sigma^{-1}).$$

There are three GL-rack homomorphisms $\mathcal{G}(\Lambda_1) \to L$, but none for $\mathcal{G}(\Lambda_2)$. $\square$

Similarly, one can show that these Legendrian knots are nonequivalent. . .

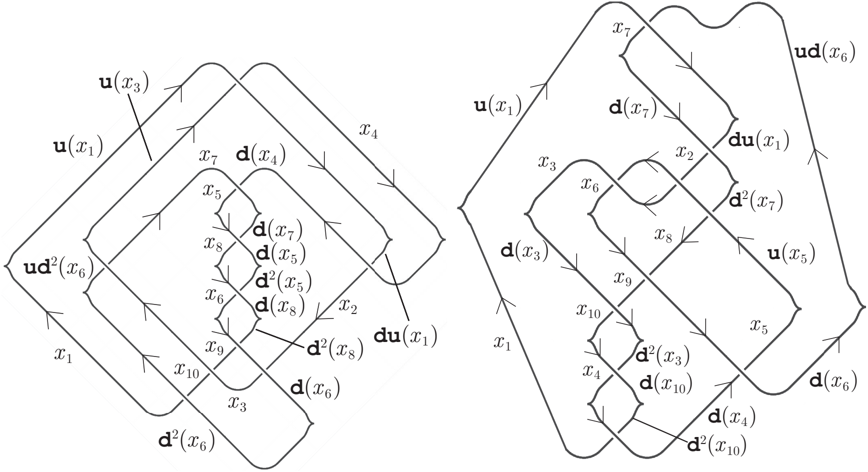


Figure: Legendrian 8_{10} knots with classical invariants $(\mathrm{tb}, \mathrm{rot}) = (-8, 3)$.

Filling out the Legendrian knot atlas (6/6)

. . . and that these Legendrian knots are also nonequivalent.

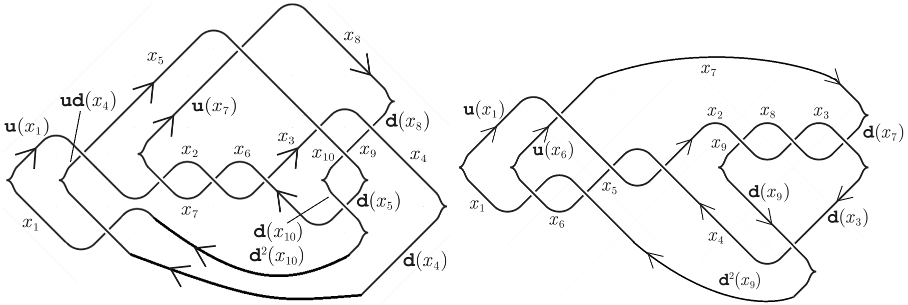


Figure: Legendrian 8_{13} knots with classical invariants $(\text{tb}, \text{rot}) = (-6, 1)$.

This settles two new conjectures of Petkova–Schwartz (2025) and completes the classification of Legendrian 8_{13} knots!

Outline

Medial racks (1/2)

Definition

A rack (X, s) is called *medial* or *abelian* if any of the following equivalent conditions hold:

- The map
$$(x, y) \mapsto s_y(x)$$
is a rack homomorphism $X \times X \to X$.
- The subgroup
$$\langle s_x s_y^{-1} \mid x, y \in X \rangle \leq \operatorname{Aut} X$$
is abelian.
- For all $x, y, z \in X$,
$$s_{s_x(z)} s_y = s_{s_x(y)} s_z.$$

Medial racks (2/2)

Example

All permutation racks and Takasaki kei are medial.

Example

A rack (X, s) is medial if and only if its *dual rack* (X, s^{-1}) is medial.

Example

Up to isomorphism, there is exactly one nonmedial rack of order 4 or lower. Let $X := \{1, 2, 3, 4\}$. In cycle notation, let

$$s_1 := \mathrm{id}_X, \quad s_2 := (34), \quad s_3 := (24), \quad \text{and } s_4 := (23).$$

Then (X, s) is a nonmedial quandle because, for example,

$$s_{s_1(3)} s_2 = (24)(34) \neq (34)(24) = s_{s_1(2)} s_3.$$

Mediality and morphisms

Due to universal algebra, we have the following.

Proposition (Grøsfjeld, 2021)

Let R and M be racks. If M is medial, then the hom-set

$$\mathrm{Hom}_{\mathsf{Rack}}(R, M)$$

has a canonical medial rack structure.

A similar result holds for GL-racks.

Remark

If Λ is a Legendrian link and M is a medial GL-rack, then the isomorphism class of the medial GL-rack

$$\mathrm{Hom}_{\mathsf{GLR}}(\mathcal{G}(\Lambda), M)$$

is a Legendrian invariant.

Tensor products of racks and GL-racks

Question

Let $\mathsf{Rack}_{\mathrm{med}}$ and $\mathsf{GLR}_{\mathrm{med}}$ be the categories of medial racks and GL-racks. Are these categories closed symmetric monoidal?

Answer

Yes—but not under the categorical/Cartesian product.

Universal algebra says that, just like with modules, **tensor products** of racks and GL-racks exist and are characterized by a universal property.

No one's really studied rack tensors at length—let's be the first!

Bihomomorphisms

Just like with bilinear maps of modules, we can consider *(GL-)rack bihomomorphisms*.

Definition

Let X, Y, Z be racks. A map $\beta_0 : X \times Y \to Z$ is a **rack bihomomorphism** if, for all $x \in X$ and $y \in Y$, the restricted maps

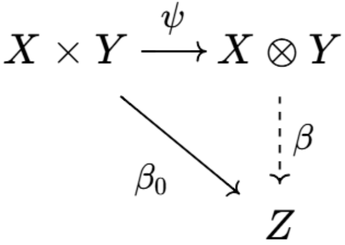
$$\beta_0(-, y) : X \to Z \quad \text{and } \beta_0(x, -) : Y \to Z$$

are rack homomorphisms. **GL-rack bihomomorphisms** are defined similarly.

Example

In A-mod, the composition of linear maps defines a bilinear map on hom-sets. Similar statements hold in Rack_{med} and GLR_{med}.

Extending the analogy with modules, **tensor products** of racks and GL-racks exist and are characterized up to isomorphism by a universal property:



Internal hom-tensor adjunctions

Thanks to universal algebra again, we have the following.

Theorem

The "medialized" rack tensor product $\otimes$ *induces a closed symmetric monoidal structure on* $\mathrm{Rack}_{\mathrm{med}}$*, and similarly for* $\mathrm{GLR}_{\mathrm{med}}$*.*

Question

How much of this structure is lost when considering nonmedial tensor products in Rack and GLR, the category of GL-racks?

We'll lose the closed structure, but what about the rest?

Tensors in “noncommutative” settings (1/2)

- Racks and groups are *noncommutative* algebraic theories.
- Universal algebra says that all algebraic theories have tensor products that satisfy a universal bihomomorphism property.
- However, they’re often quite pathological in noncommutative algebraic theories.

Tensors in "noncommutative" settings (2/2)

Example

The universal-algebraic tensor product of groups is isomorphic to the usual $\mathbb{Z}$-module tensor product of their abelianizations:

$$G \otimes H \cong G^{\mathrm{ab}} \otimes_{\mathbb{Z}} H^{\mathrm{ab}}.$$

In particular,

- there is no tensor unit, and
- all tensor products are abelian.

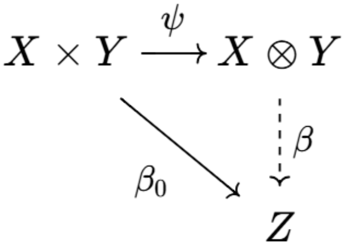
Question

Do tensor products of racks and GL-racks face similar limitations?

Existence of a tensor unit (1/3)

Theorem

In Rack *and* GLR, *the free objects on one generator are tensor units.*



Proof sketch (1/3).

1. For the free rack F on one element, it suffices to show that F satisfies the universal property of $R \otimes F$.
   - We need to construct ψ and β given β_0.

Existence of a tensor unit (2/3)

Proof sketch (2/3).

2. Identify $F \cong (\mathbb{Z}, \tau)_{\mathrm{perm}}$, and show that the map $\psi : R \times \mathbb{Z} \to X$ defined by
$$(x, k) \mapsto \theta_R^k(x)$$
is a bihomomorphism. *(Use the fact that* $\theta^k \in Z(\mathrm{Rack})$*.)*
3. Given a bihomomorphism $\beta_0 : R \times \mathbb{Z} \to S$, define $\beta : X \to Y$ to be the restriction
$$x \mapsto \beta_0(x, 0).$$
4. Check commutativity. Uniqueness follows from the surjectivity of ψ, completing the proof for racks.

□

Existence of a tensor unit (3/3)

Proof sketch (3/3).

5. For GL-racks, identify $F = ((\mathbb{Z}^2, \sigma)_{\mathrm{perm}}, \mathrm{u})$, and verify the universal property with

$$\psi(x, m, n) := \mathrm{u}^n \theta_R^m(x) \quad \text{and } \beta(x) := \beta_0(x, 0, 0).$$

□

Corollary

Tensor products of racks are not necessarily medial, even if one of the tensor factors is; and similarly for tensor products of GL-racks.

Outline

The group of GL-structures

The definition of GL-racks given in 2023 was much longer than our equivalent one. This made the original authors pose the following question.

Question (Karmakar–Saraf–Singh, 2023)

Given a rack R, what is the set U_R of all possible GL-structures on R?

To answer this, we can just rephrase our new definition.

Answer

- U_R is the centralizer

$$U_R = C_{\mathrm{Aut}\,R}(\mathrm{Inn}\,R) \trianglelefteq \mathrm{Aut}\,R,$$

  where $\mathrm{Inn}\,R$ is the (normal) subgroup of $\mathrm{Aut}\,R$ generated by the s_x's.
- $(R, \mathrm{u}_1) \cong (R, \mathrm{u}_2)$ if and only if u_1 and u_2 are conjugate in $\mathrm{Aut}\,R$.

Classifying results (1/3)

So, classifying GL-structures reduces to a group theory exercise for certain infinite families of racks!

Proposition (Permutation GL-racks)

Let $P = (X, \sigma)_{\mathrm{perm}}$ be a permutation rack. Then

$$U_P = C_{S_X}(\sigma) = \mathrm{Aut}\, P,$$

and $U_P/\sim$ is the set of conjugacy classes of $C_{S_X}(\sigma)$.

Proposition (Conjugation GL-quandles)

Let G be a group, and let $Q := \mathrm{Conj}\, G$.

- *If G is abelian, then $U_Q = S_G$, and $U_Q/\sim$ is the set of conjugacy classes of S_G.*
- *On the other hand, if G is centerless, then $U_Q = \{\mathrm{id}_G\}$.*

Classifying results (2/3)

Proposition

If A is an abelian group without 2-torsion, then the only GL-structure on the Takasaki kei $T(A)$ is id_A.

Proof sketch.

For all such Takasaki kei, the automorphism group G and inner automorphism group H are known to be certain semidirect products. Use this classification to show that

$$U_{T(A)} = C_G(H) \subseteq \{(0, \psi) \in G : \psi|_{2A} = \mathrm{id}_{2A}\} = \{\mathrm{id}_A\}.$$

□

Classifying results (3/3)

Definition

For all $n \in \mathbb{N}$, the Takasaki kei $R_n := T(\mathbb{Z}/n\mathbb{Z})$ is a **dihedral quandle**.

Proposition

For all even $n \geq 2$,

$$U_{R_n} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } 4 \nmid n, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } 4 \mid n. \end{cases}$$

If $4 \nmid n$, then the two GL-structures on U_{R_n} yield nonisomorphic GL-quandles. If $4 \mid n$, there's exactly one isomorphic pair.

Proof sketch.

Once again, $\operatorname{Aut} R_n$ and $\operatorname{Inn} R_n$ are certain semidirect products. Compute how conjugation works in these semidirect products to compute the centralizer of $\operatorname{Inn} R_n$ in $\operatorname{Aut} R_n$. □

Automorphism groups

Similarly, we rephrase the definition of a GL-rack automorphism as follows.

Proposition

For all GL-racks $(R, \mathtt{u})$*,*

$$\mathrm{Aut}_{\mathrm{GLR}}(R, \mathtt{u}) = C_{\mathrm{Aut}\, R}(\mathtt{u}).$$

Example

For all dihedral GL-quandles $(R_n, \mathtt{u})$,

$$\mathrm{Aut}_{\mathrm{GLR}}(R_n, \mathtt{u}) \cong \mathbb{Z}/n\mathbb{Z} \rtimes (\mathbb{Z}/n\mathbb{Z})^\times$$

except for a certain GL-structure $\mathtt{u}'$ on R_n when $4 \mid n$, in which case

$$\mathrm{Aut}_{\mathrm{GLR}}(R_n, \mathtt{u}') \cong 2\mathbb{Z}/n\mathbb{Z} \rtimes (\mathbb{Z}/n\mathbb{Z})^\times.$$

Categorical centers

The following example allows us to compute $Z(\text{GLR})$.

Example

The automorphism group of the free GL-rack on one element is $\mathbb{Z}^2$.

Recall from earlier: The **center** of a category $\mathcal{C}$ is the commutative monoid $Z(\mathcal{C})$ of natural endomorphisms of the identity functor $\mathbf{1}_{\mathcal{C}}$.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Y & \xrightarrow{\eta_Y} & Y \end{array}$$

Essentially, elements of $Z(\mathcal{C})$ are collections η of morphisms in $\mathcal{C}$ that commute with all other morphisms φ in $\mathcal{C}$.

Results

Theorem

Let Θ be the collection of θ_R's for all racks R, and let $\mathtt{u}$ be the collection of all GL-structures on racks. Then:

1. $Z(\mathsf{GLR})$ *is the free abelian group* $\langle \Theta, \mathtt{u} \rangle \cong \mathbb{Z}^2$.
2. *The categorical centers of GL-quandles and Legendrian racks are each the free group* $\langle \mathtt{u} \rangle \cong \mathbb{Z}$.
3. *The categorical center of Legendrian quandles is the group* $\langle \mathtt{u} \mid \mathtt{u}^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Proof sketch.

Each of these categories is *strongly generated* by the free object F on one element. Show that the center is determined by F. Then, relate each group in the claim to the automorphism group of F. □

Outline

Isomorphism of categories

Recall:

- A functor $F : \mathcal{C} \to \mathcal{D}$ is an **equivalence of categories** if there's another functor $G : \mathcal{D} \to \mathcal{C}$ such that FG and GF are *naturally isomorphic* to the identity functors $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$, respectively.
- If these natural isomorphisms are actually equalities, then we say that F is an isomorphism of categories.

Example

The category of representations of a group G on a field $\mathbb{F}$ is isomorphic to the category of left $\mathbb{F}[G]$-modules.

Construction of functors

Let GLQ be the category of GL-quandles. We will show that Rack and GLQ are isomorphic. Define a functor F : Rack $\rightarrow$ GLQ by

$$R = (X, s) \mapsto (X, \theta_R^{-1} s, \theta_R),$$

and define a functor G : GLQ $\rightarrow$ Rack by

$$(X, s, \mathrm{u}) \mapsto (X, \mathrm{u}s).$$

They'll fix (homo)morphisms as set maps.

Result

Theorem

The categories Rack *and* GLQ *are equivalent—in fact, isomorphic.*

Proof.

After verifying that F and G are (covariant) functors, we simply verify that $FG = \mathbf{1}_{\mathsf{GLQ}}$ and $GF = \mathbf{1}_{\mathsf{Rack}}$. □

Remark

This is cool because of how independently racks and GL-quandles are defined—that these categories are isomorphic is not obvious from the definitions!

Outline

Questions for future research

- Fill out more of the Legendrian knot atlases!
- Contact-geometric characterization of $\mathcal{G}(\Lambda)$?
- Rack-theoretic invariants of virtual Legendrian links or transverse links?
- More interesting properties of rack tensor products?
- Enriched category theory applied to racks?
- Classify more families of GL-racks!

Thank you!

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