MATH 370 (Sp. 2025): ULA Small Group Session #2 with Lực Ta

As requested, let's do a review of normal subgroups. Then, we'll do a couple problems to practice using normal subgroups in a Galois theory context :) But first, remember to sign in, using either the QR code or this link.

Definition. Let $H \leq G$. There are many equivalent definitions of what it means for H to be a *normal subgroup* of G, in which case we write $H \trianglelefteq G$. Here's what I think the most useful ones are:

1. There exists a group K and a group homomorphism $\varphi : G \to K$ such that $\ker \varphi = H$.
2. For all $g \in G$ and $h \in H$, we have $ghg^{-1} \in H$.
3. For all $g \in G$, we have $gHg^{-1} \subseteq H$.
4. For all $g \in G$, we have $gH = Hg$.
5. Multiplication of left cosets $g_1H \cdot g_2H = (g_1g_2)H$ is well-defined and induces a *quotient group* structure on the set $G/H = \{gH \mid g \in G, \text{ and } g_1H = g_2H \text{ iff } g_2 = g_1h \text{ for some } h \in H\}$.
6. H is a union of conjugacy classes in G.

We also have some sufficient (but not necessary) conditions for a subgroup $H \leq G$ to be normal.

Proposition. *If any of the following are true, then $H \trianglelefteq G$.*

1. *G is abelian.*
2. *G is finite, and $[G : H]$ is the smallest prime number dividing $|G|$. (Recall that $[G : H] = |G|/|H|$ is called the* index *of H in G.) In particular, any subgroup of index 2 is normal.*
3. *G is the (internal) direct product of H and some other subgroup of G.*

Let's get a little practice with normal subgroups.

Problem 1. Let $D_n = \langle r, s \mid r^n = s^2 = 1,\ srs = r^{-1} \rangle$ be the dihedral group of order $2n$. Consider the subgroups $\langle r \rangle$ and $\langle s \rangle$. Are either of them normal in D_n?

Problem 2. Consider the *quaternion group* $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, where multiplication is defined as follows:

$$ij = k, \quad jk = i, \quad ki = j, \quad i^2 = j^2 = k^2 = -1, \quad (-1)^2 = 1, \quad \text{and } (-1)g = -g = g(-1) \quad \forall g \in Q_8.$$

(Note that Q_8 is nonabelian.) Describe the subgroups of Q_8. Which ones are normal?

Problem 3. Given a subgroup $H \leq G$, define the *centralizer* of H in G, denoted by $C_G(H)$, to be the set of all elements of G that commute with every element of H. That is,

$$C_G(H) = \{g \in G \mid ghg^{-1} = h\}.$$

Show that if $H \trianglelefteq G$, then $C_G(H) \trianglelefteq G$. (In particular, the *center* of G, defined as $Z(G) := C_G(G)$, is normal in G.)

Problem 4. (This one's just for fun.) Let $n \in \mathbb{Z}^+$ be a positive integer, and let $\mathbb{F}$ be a field. Consider the *general linear group* $\mathrm{GL}_n(\mathbb{F})$, which is the group of invertible $n \times n$ matrices with entries in $\mathbb{F}$. Also, consider the *special linear group*

$$\mathrm{SL}_n(\mathbb{F}) = \{M \in \mathrm{GL}_n(\mathbb{F}) \mid \det M = 1\}.$$

Show that $\mathrm{SL}_n(\mathbb{F})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{F})$. What group is $\mathrm{GL}_n(\mathbb{F})/\mathrm{SL}_n(\mathbb{F})$ isomorphic to? *(Hint: Show that $\mathrm{SL}_n(\mathbb{F})$ is the kernel of a certain homomorphism from $\mathrm{GL}_n(\mathbb{F})$ to $\mathbb{F}^\times$.)*

Normal subgroups are crucial for computing semidirect products of groups.

Theorem ("Internal semidirect product theorem" or "identification theorem," DF p. 180)**.** *Let G be any group, and let H and K be subgroups of G such that*

1. $H \trianglelefteq G$,
2. $H \cap K = 1$, *and*
3. $|G| = |H||K|$.

Let $\varphi : K \to \operatorname{Aut}(H)$ be the conjugation map $k \mapsto [h \mapsto khk^{-1}]$. Then G is isomorphic to the (internal) semidirect product $H \rtimes_\varphi K$.

Proposition (DF p. 177)**.** *Let G be the semidirect product $H \rtimes K$. Then $H \trianglelefteq G$. Moreover, the following are equivalent:*

1. *G is isomorphic to the direct product $H \times K$.*
2. *$\varphi : K \to \operatorname{Aut}(H)$ is the trivial homomorphism.*
3. *K acts trivially on H. That is, $k \cdot h = h$ for all $k \in K$ and $h \in H$.*
4. $K \trianglelefteq G$.

Problem 5. Deduce from Problem 2 that Q_8 can't be written as an (internal) semidirect product $H \rtimes K$ with $1 < |H|, |K| < 8$.

Finally, let's see what implications normal subgroups have for Galois theory.

Problem 6. Use the internal semidirect product theorem to classify the Galois groups of the following polynomials $f \in \mathbb{Q}[x]$ over $\mathbb{Q}$.

(a) $f = x^3 - 2$.

(b) $f = x^4 - 2$.

(c) $f = x^7 - 13$.

Problem 7. Let's apply what the fundamental theorem of Galois theory tells us about normal subgroups and quotient groups.[1] Let $\alpha := \sqrt{(2+\sqrt{2})(3+\sqrt{3})}$, and let $K = \mathbb{Q}(\alpha)$. You can take for granted that $\operatorname{Gal}(K/\mathbb{Q}) \cong Q_8$.

(a) Let $H \leq \operatorname{Gal}(K/\mathbb{Q})$. Is $K^H/\mathbb{Q}$ necessarily Galois? (Use Problem 2.)

(b) You can take for granted that $\beta := \sqrt{2}+\sqrt{3} \in K$, and $[K : \mathbb{Q}(\beta)] = 2$. Use the previous part to give a new proof that $\operatorname{Gal}(\mathbb{Q}(\sqrt{2}+\sqrt{3})/\mathbb{Q})$ is isomorphic to $Z_2 \times Z_2$. This new proof should not refer at all to polynomials or roots, and it should not use the fact that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

You're doing great! :)

[1] In particular, see the discussion under "Properties of The Galois Correspondence" in Din's notes.