MATH 370 (Sp. 2025): ULA Exam II Review Session (with Lực Ta and Adam Wesley)

Remember to sign in, using either the QR code or this link.

Problem 1. Let K/F be a finite Galois extension, and let $f \in F[x]$ be a separable polynomial of degree n. Show that if K is the splitting field of f over F, then $[K : F] \mid n!$.

Problem 2. Use the internal semidirect product theorem to classify the Galois groups of the following polynomials $f \in \mathbb{Q}[x]$ over $\mathbb{Q}$.

(a) $f = x^3 - 2$.
(b) $f = x^4 - 2$.
(c) $f = x^7 - 13$.
(d) $f = (x^3 - 2)(x^2 - 2)(x^2 + 1)$.

Problem 3. This problem gives us a chance to practice (a) identifying normal subgroups of a given group and (b) applying what the fundamental theorem of Galois theory tells us about normal subgroups and quotient groups.[1] Let $\alpha := \sqrt{(2+\sqrt{2})(3+\sqrt{3})}$. You can take for granted that $G := \mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ is isomorphic to the group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, where multiplication is defined as follows:

$$ij = k, \quad jk = i, \quad ki = j, \quad i^2 = j^2 = k^2 = -1, \quad (-1)^2 = 1, \quad \text{and } (-1)g = -g = g(-1) \quad \forall g \in G.$$

(This group is called the *quaternion group*.)

(a) Let $H \leq G$. In the context of this problem, is $\mathbb{Q}(\alpha)^H/\mathbb{Q}$ necessarily Galois? If so, then what is $\mathrm{Gal}(\mathbb{Q}(\alpha)^H/\mathbb{Q})$ isomorphic to?
(b) Let $\beta := \sqrt{2} + \sqrt{3}$. You can take for granted that $\beta \in \mathbb{Q}(\alpha)$, and $[\mathbb{Q}(\alpha) : \mathbb{Q}(\beta)] = 2$. Combine these facts with part (a) to give a new proof that $\mathrm{Gal}(\mathbb{Q}(\beta)/\mathbb{Q}) \cong Z_2 \times Z_2$.

Problem 4. Let p be prime, and let $f \in \mathbb{Q}[x]$ be irreducible of degree p. Prove that if f has exactly two nonreal roots, then the Galois group of f over $\mathbb{Q}$ is isomorphic to S_p.

(Hint: What can we say about the action of the Galois group on the roots? Also, what do we know about S_p when p is prime?)

The next two problems offer practice with purely inseparable field extensions. Recall from hw6 that a field extension K/F is purely inseparable if and only if, for every element $\alpha \in K$, there exists a positive integer $n > 0$ such that $\alpha^{p^n} \in F$, where p is the characteristic of the extension.

Problem 5. Let $K/M/F$ be field extensions. Show that K/F is purely inseparable if and only if K/M and M/F are both purely inseparable.

Problem 6. Let $K/M/F$ be field extensions of finite degree such that K/M is normal and M/F is purely inseparable. Prove that K/F is a normal field extension. Deduce that all purely inseparable field extensions of finite degree are normal.

Problem 7. True/false. Justify your answers.[2]

(a) Any field extension of degree n contains an element of degree n (over the base field).

[1] In particular, see the discussion under "Properties of The Galois Correspondence" in Din's notes.
[2] Credit to Hamilton Wan for several of these questions.

(b) The Galois group of an inseparable extension is trivial.

(c) The Galois group of a purely inseparable extension is trivial.

(d) Every purely inseparable extension is inseparable.

(e) If a polynomial $f \in F[x]$ splits over some $K \supset F$, then K/F is normal.

(f) A reducible polynomial of degree 5 is solvable by radicals.

(g) A reducible polynomial of degree 6 is solvable by radicals.

(h) A reducible polynomial of degree 6 that has no roots in the base field is solvable by radicals.

Problem 8. *(Challenge problem A.)* Let K/F and L/F be finite Galois extensions, and let KL/F be the smallest extension of F that contains K and L. Use the internal (semi)direct product theorem to show that if $K \cap L = F$, then there is an isomorphism $\mathrm{Gal}(KL/F) \cong \mathrm{Gal}(K/F) \times \mathrm{Gal}(L/F)$.

(Hint: The setup of this problem might remind you of a previous homework problem. One approach might be to revisit your proof of the previous homework problem to find subgroups of $\mathrm{Gal}(KL/F)$ *that are isomorphic to* $\mathrm{Gal}(K/F)$ *and* $\mathrm{Gal}(L/F)$*. Then, show that the internal (semi)direct product theorem applies to these subgroups. Along the way, you might use the fact that* KL/F *is Galois, which requires some justification.)*

Problem 9. *(Challenge problem B.)* Let $\mathbb{Q}[x, y]$ be the ring of polynomials in two variables with rational coefficients. Show that

$$\mathrm{Der}_{\mathbb{Q}}\, \mathbb{Q}[x, y] = \left\{ f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} : f, g \in \mathbb{Q}[x, y] \right\}.$$

(Hint: For the "$\supseteq$*" direction, use the fact that* $\mathrm{Der}_{\mathbb{Q}}\, \mathbb{Q}[x, y]$ *is a* $\mathbb{Q}[x, y]$*-module. For the "*$\subseteq$*" direction, start by arguing that all elements* $D \in \mathrm{Der}_{\mathbb{Q}}\, \mathbb{Q}[x, y]$ *are* $\mathbb{Q}$*-linear maps. Then, use this fact to reduce the problem to computing what* $D(x^m y^n)$ *is, where* $m, n \in \mathbb{Z}_{\geq 0}$ *are nonnegative integers.)*

Problem 10. *(Challenge problem C, practice with solvable groups in general)* Prove that every group of order $n < 60$ is solvable. In particular, this proves that A_5 is the smallest group which is not solvable. *(Hint: lots of these orders admit only abelian groups, which are solvable. Only one of these orders has three distinct prime factors* $(30 = 2 \times 3 \times 5)$*. The Sylow theorems are your friend.)*

You're doing great! :)