MATH 370 (Sp. 2025): ULA Midterm I Review Session (with Lực Ta and Adam Wesley)

Remember to sign in, using either the QR code or this link.

Problem 1. Let F be a subfield of $\mathbb{C}$, and let K/F be a degree 2 extension. Is K/F necessarily Galois?

Problem 2. Let $F \subset M \subset K$ be fields.

(a) Suppose K/F is Galois. Is K/M necessarily Galois?
(b) Suppose K/F is Galois. Is M/F necessarily Galois?
(c) Suppose M/F and K/M are both Galois. Is K/F necessarily Galois?

Problem 3. Classify the Galois groups of the following polynomials.

(a) $f(x) := x^3 - 3x + 1$ over $\mathbb{Q}$.
(b) The minimal polynomial of $\sqrt{2+i}$ over $\mathbb{Q}$.
(c) The minimal polynomial of $\sqrt{2+\sqrt{2}}$ over $\mathbb{Q}$.
(d) $f(x) := x^4 - 2$ over F, where F is the splitting field of $x^2 - 2$ over $\mathbb{Q}$.
(e) The same polynomial as in the last part, but now over $\mathbb{Q}$.

Problem 4. Let K be a subfield of $\mathbb{R}$, and let $f \in K[x]$ be an irreducible polynomial. Show that if the Galois group of f has odd order, then the discriminant of f is positive. *(Hint: Trying to prove that an abstract polynomial has a positive discriminant is kind of weird. Is there a way to avoid doing that while still solving the problem?)*

Problem 5. Let K/F be a Galois extension such that $\mathrm{Gal}(K/F) \cong Z_3 \times Z_{18}$. How many intermediate fields M are there such that

(a) $[M : F] = 18$
(b) $[M : F] = 27$
(c) $[M : F] = 3$
(d) $[M : F] = 6$
(e) $\mathrm{Gal}(K/M) \cong Z_2$
(f) $|\mathrm{Gal}(K/M)| = 6$
(g) $\mathrm{Gal}(K/M) \cong Z_{27}$
(h) $|\mathrm{Gal}(K/M)| = 27$

Problem 6. True or false? Justify your answer.

(a) If $\alpha \neq \beta$ are both irrational, then $\mathbb{Q}(\alpha, \beta)$ is not a simple extension of $\mathbb{Q}$.

(b) Every algebraic extension is finite.

(c) Two extensions of the same degree are isomorphic.

(d) Suppose there exist α and β such that the extensions $\mathbb{Q}(\alpha)/\mathbb{Q}$ and $\mathbb{Q}(\beta)/\mathbb{Q}$ are isomorphic. Then α and β have the same minimal polynomial over $\mathbb{Q}$.

Problem 7. Let $K/\mathbb{Q}$ be a Galois extension of degree 4 and suppose that $i \in K$. Prove that $\mathrm{Gal}(K/\mathbb{Q}) \simeq Z_2 \times Z_2$. *Hint: what can you say about the extension $K/\mathbb{Q}(i)$?*

Problem 8. Show that there are infinitely many irreducible polynomials over any field. *Hint: think about Euclid's proof that there are infinitely many primes in $\mathbb{Z}$.*

You're doing great! :)