MATH 370 (Sp. 2025): ULA Final Review Session (with Lực Ta and Adam Wesley)

Remember to sign in, using either the QR code or this link.

Problem 1. Let F be a finite field, and let K/F be an extension of finite degree.

(a) Show that K/F is Galois. *(For practice, give two different proofs, one using a homework result and the other using only the finite fields theorem.)*

(b) Let p be prime, and let $n, m \geq 1$ be integers such that $m \mid n$. What is $[\mathbb{F}_{p^n} : \mathbb{F}_{p^m}]$? *(For practice, give two different justifications, one using the definition of the degree of a field extension and the other combining part (b) with a homework result.)*

(c) *Optional:* Show that K/F is simple.

Problem 2. Show that if F is an algebraically closed field, then $|F| = \infty$. *(For practice, give two different proofs, one using a homework result and the other using only the finite fields theorem.)*

Problem 3. Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Is $\overline{\mathbb{Q}}$ countable or uncountable?

Problem 4. Show that if K/F is a finite extension, then $[K : F]$ is divisible by $|\operatorname{Gal}(K/F)|$. Deduce that if $\alpha \in \mathbb{C}$ is constructible over $\mathbb{Q}$, then $\operatorname{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ is solvable.

Problem 5.

(a) Let $K/\mathbb{Q}$ be a finite Galois extension. Describe a one-to-one correspondence between

  (i) subextensions $K/M/\mathbb{Q}$ such that $[M : \mathbb{Q}] = 2$ and
  (ii) nontrivial group homomorphisms from $\operatorname{Gal}(K/\mathbb{Q})$ to $\mathbb{Z}/2\mathbb{Z}$.

*(Note: This does **not** generalize to integers other than 2.)*

(b) Now, let K be the splitting field of the 60th cyclotomic polynomial Φ_{60} over $\mathbb{Q}$.

  (i) How many subextensions $K/M/\mathbb{Q}$ are there such that $[M : \mathbb{Q}] = 8$?
  (ii) Using part (a) (or otherwise), find the number of subextensions $K/M/\mathbb{Q}$ such that $[K : M] = 8$.

Problem 6. True or false? Justify your answer. Where possible, try to produce an example/counterexample. If the statement is false, what needs to change to make it true (excluding things like inserting the word “not")?

(a) A purely inseparable extension is inseparable.

(b) There exists a nontrivial extension of $\mathbb{R}$ not contained in $\mathbb{C}$.

(c) There exists a nontrivial algebraic extension of $\overline{\mathbb{Q}}$.

(d) There are no irreducible polynomials in $\mathbb{C}[x]$.

Problem 7. What groups arise as Galois groups of quartic polynomials over $\mathbb{Q}$? For each isomorphism type G, list a polynomial with Galois group isomorphic to G. *Note: (i) Remember that these polynomials may be reducible. (ii) If you're having trouble thinking of examples for some of these, try taking some from Problem 10.*

Problem 8. Prove that the polynomial $f(x) = x^5 - x + 15 \in \mathbb{Q}[x]$ is solvable by radicals.

Problem 9. Prove that a field F is algebraically closed iff every endomorphism (i.e., linear map) $F^n \to F^n$ has an eigenvector. *Linear algebra is useful!*

Problem 10.

(a) Show that if $f = x^4 + rx + s \in F[x]$ factors into two quadratics $f = (x^2 + a_1 x + b)(x^2 + a_2 x + c)$ over a field K containing F, then $a_1 = -a_2$.

(b) Compute the Galois groups of the following quartics over $\mathbb{Q}$. You may use the discriminant formula $\Delta(x^4 + rx + s) = -27r^4 + 256s^3$ and the resolvent cubic formula $g = x^3 - 4sx - r^2$.

   (i) $f = x^4 + x^3 + x^2 + x + 1$.

   (ii) $f = x^4 + 1$.

   (iii) $f = x^4 - 4x + 2$.

   (iv) $f = x^4 + 8x + 12$. *(Feel free to use a calculator for the discriminant for this one.)*

   (v) $f = x^4 + 3x + 3$.

   (vi) $f = x^4 + 6x^3 + 13x^2 + 12x + 4$.

Problem 11. Let $n \geq 1$ be an integer. Prove that for all groups G of order n, there exists a subfield F of the field of rational functions $K := \mathbb{Q}(x_1, \ldots, x_n)$ such that $\mathrm{Gal}(K/F) \cong G$.

You're doing great! :)