MATH 305 (SP24)
Midterm Review Session with Luc :)

Throughout this review session, I'll use m to denote the Lebesgue measure, $\mathcal{M}$ to denote the Lebesgue σ-algebra, and $\mathcal{B}$ to denote the Borel σ-algebra.

Problem 1. Let μ and ν be finite measures on the measure space $(X, \mathcal{A})$, and let $\mathcal{C} \subset \mathcal{A}$. Suppose that for all $A \in \mathcal{C}$, we have $\mu(A) = \nu(A)$. Must $\mu = \nu$?

Problem 2. Is every Lebesgue measurable set is the union of a Borel set and a Lebesgue null set?

Problem 3. More generally, suppose $(X, \mathcal{A}, \mu)$ is a measure space, let $\mathcal{N}$ be the collection of μ^*-null subsets of X, and let $\mathcal{C} = \sigma(\mathcal{A} \cup \mathcal{N})$. Suppose also that for all $N \in \mathcal{N}$, there exists some $M \in \mathcal{A}$ such that $N \subset M$, and $\mu(M) = 0$.

(a) Show that $D \in \mathcal{C}$ if and only if there exist $A_D \in \mathcal{A}$ and $N_D \in \mathcal{N}$ such that $D = A_D \cup N_D$.
(b) So, for all $D \in \mathcal{C}$, we can define $\overline{\mu}(D) = \mu(A_D)$. Assuming that $\overline{\mu}$ is well-defined, show that $\overline{\mu}$ defines a measure on $\mathcal{C}$. (We say that $(X, \mathcal{C}, \overline{\mu})$ is the *completion* of $(X, \mathcal{A}, \mu)$.)

Problem 4. True or false:

(a) There exists a nonempty open subset $U \subset \mathbb{R}$ such that $m(U) = 0$.
(b) If $E \subset \mathbb{R}$ and $\operatorname{int}(E) = \emptyset$, then $m^*(E) = 0$.
(c) If $E \subset \mathbb{R}$ and $m^*(E) = 0$, then $\operatorname{int}(E) = \emptyset$.

Problem 5. Suppose $(\mathbb{R}, \mathcal{M}, \varphi)$ is a measure space such that

(i) $\varphi[-\pi/2, 1 - \pi/2] = e$, and
(ii) for all $A \in \mathcal{M}$ and for all $r \in \mathbb{R}$, we have $\varphi(A + r) = \varphi(A)$.

Compute $\varphi(\mathbb{Q}^c)$.

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$, and let $E = \{x \in \mathbb{R} \mid f \text{ isn't continuous at } x\}$. True or false:

(a) If $m(E) = 0$, then f is Lebesgue measurable.
(b) If f is Lebesgue measurable, then $m(E) = 0$.
(c) If $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $E = \emptyset$, then $g \circ f$ is Lebesgue measurable.
(d) If $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $E = \emptyset$, then $f \circ g$ is Lebesgue measurable.
(e) There exists a non-Lebesgue measurable, non-negative function $h : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that the function $\sqrt{h}$ is Lebesgue measurable.
(f) If f is differentiable, then f' is Lebesgue measurable.

Problem 7. Is every monotonic function $f : \mathbb{R} \to \mathbb{R}$ **Borel** measurable?

Problem 8. Construct a Lebesgue measurable subset of $\mathbb{R}$ that isn't Borel.

Problem 9. Let $f : [0, 1] \to \mathbb{R}$. Suppose that for all $r \in \mathbb{R}$, the set $f^{-1}(\{r\})$ is Lebesgue measurable. Must f be Lebesgue measurable?

You're doing great! Good luck on the midterm—I believe in you! :)