MATH 305 (SP24)
Reading Period Office Hours with Luc :)

Problem 1. Let $E \subset \mathbb{R}$ be a measurable subset. Suppose $f \in L^1(E)$ is bounded. Is it true that $f \in L^2(E)$?

Problem 2. Let $E \subset \mathbb{R}$ be a measurable subset such that $m(E) < \infty$. Is it true that $L^2(E) \subset L^1(E)$?

Problem 3. Construct functions $f \in L^1(\mathbb{R}) \setminus L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$.

Problem 4. Suppose $f_n \to f$ and $g_n \to g$ in $L^2(\mathbb{R})$. Is it true that $\lim_{n\to\infty}\langle f_n, g_n\rangle = \langle f, g\rangle$?

Problem 5. Let $f : \mathbb{R} \to [0, \infty)$ be a non-negative measurable function. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} n \ln\left(1 + \frac{f(x)}{n}\right) dx = \int_{\mathbb{R}} f(x)\, dx.$$

(Feel free to use the fact from Math 255 that for all $c \geq 0$, the sequence $(1 + c/n)^n$ is increasing.)

Problem 6. Let f and g be functions with convergent Fourier series. Show that the sum of their Fourier series is the Fourier series of $f + g$.

Problem 7. Write the Fourier series of the **sign function** (also called the **signum function**)

$$\operatorname{sgn}(x) := \begin{cases} -1 & x \in [-\pi, 0) \\ 0 & x = 0 \\ 1 & x \in (0, \pi] \end{cases}.$$

Problem 8. Let $a \in \mathbb{C}$ such that $|a| < 1$. Fix $k \in \mathbb{Z}^+$. Compute the Fourier coefficients of

$$f(x) := \frac{1}{1 - ae^{kix}}.$$

Problem 9. Define $f(x) := x^3 - \pi^2 x$. The Fourier series of f is $\sum_{N\in\mathbb{Z}, N\neq 0} 6e^{iNx}/N^3$. (Feel free to verify that if you want, but it's a pain.) Use this to show that $\sum_{n=1}^{\infty} 1/n^6 = \pi^6/945$.

Problem 10. Let $f \in C[-\pi, \pi]$ be a differentiable periodic function with a continuous derivative f'. Compute the Fourier coefficients $\widehat{f'}(n)$ of f', writing them in terms of $\widehat{f}(n)$.

Problem 11. Let $f \in L^1(\mathbb{R})$ be a continuous function. Without using the fundamental theorem of calculus, show that Lebesgue's differentiation theorem holds for f at *all* points (not just a.e.).

Problem 12. Let $f \in L^1(\mathbb{R})$, and consider the **Fourier transform** $\mathcal{F} : \mathbb{R} \to \mathbb{C}$ of f:

$$\mathcal{F}(\omega) := \int_{\mathbb{R}} f(x) e^{-ix\omega}\, dx.$$

Prove that $\mathcal{F}$ is continuous. *(Hint: In a metric space, continuity is a statement about limits. Limits and integrals... Hmm...)*

You've been doing amazing work! I'm so proud of you all for all the growth you've made as mathematicians this semester, and you should be too :) It was a real pleasure working with you this spring. Hopefully we'll cross paths again, but even if we don't, remember that I'm always an email away if you ever need any support. Good luck on the final—I believe in you!