COUNTEREXAMPLE ABOUT LIE RACKS

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The purpose of this writeup is to propose infinitely many pairwise nondiffeomorphic counterexamples to the following theorem.

Theorem 1 ([1, Thm. 4.6])**.** *Let $(M, \triangleright, \mathbf{0})$ be a Lie rack integrating a Leibniz algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is a Lie algebra if and only if X is a quandle. In that case, $(M, \triangleright, \mathbf{0})$ is isomorphic to the conjugation quandle of a Lie group integrating $\mathfrak{g}$.*

For every integer $n \geq 2$, we construct a Lie rack $(M, \triangleright, \mathbf{0})$ that integrates a Lie algebra $\mathfrak{g} = (T_{\mathbf{0}}M, [\cdot, \cdot])$ despite not being a quandle. The underlying manifold M is the disjoint union of n copies of the real line, that is,

$$M := \mathbb{R} \times \{0, 1, \dots, n-1\}$$

equipped with its canonical manifold structure. Pick the basepoint of M to be $\mathbf{0} \in M$, and define a (left-distributive) rack operation $\triangleright$ on M by

$$(x, k) \triangleright (y, \ell) := \begin{cases} (y, \ell) & \text{if } k = 0, \\ (-y, \ell) & \text{if } k \geq 1. \end{cases}$$

Checking the following is straightforward.

Proposition 2. *The triple $(M, \triangleright, \mathbf{0})$ is a Lie rack that is not a quandle.*

On the other hand, we also have the following.

Proposition 3. *Let $\mathfrak{g} = (T_{\mathbf{0}}M, [\cdot, \cdot])$ be the (left) Leibniz algebra associated to $(M, \triangleright, \mathbf{0})$. Then $\mathfrak{g}$ is abelian. In particular, $\mathfrak{g}$ is a Lie algebra, so Theorem 1 is false.*

Proof. We have to show that $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. By definition,

$$[X, Y] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\gamma(t)}(X) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} ((\mathrm{d}L_{\gamma(t)})_{\mathbf{0}}(X)),$$

where $\gamma\colon [0, 1] \to M$ is a smooth path such that $\gamma(0) = \mathbf{0}$ and $\gamma'(0) = Y$.

Since γ is continuous, its image is contained in the same connected component of M as $\mathbf{0}$. In other words, for all $t \in [0, 1]$, we have $\gamma(t) \in \mathbb{R} \times \{0\}$, so $L_{\gamma(t)} = \mathrm{id}_M$. Therefore, $(\mathrm{d}L_{\gamma(t)})_{\mathbf{0}} = \mathrm{id}_{\mathfrak{g}}$, so $[X, Y] = 0$. □

Remark 4. We can generalize this family of counterexamples by replacing $\mathbb{R}$ with any manifold X, replacing the map $y \mapsto -y$ with any nontrivial involutory diffeomorphism $f\colon X \to X$, and replacing $\mathbf{0}$ with any fixed point of f. In particular, Riemannian symmetric spaces provide infinitely many counterexamples to Theorem 1.

References

[1] G. La Rosa and M. Mancini, *Two-step nilpotent Leibniz algebras*, Linear Algebra Appl. **637** (2022), 119–137. MR4355961