LOCAL INTEGRATION OF COMPATIBLE LIE ALGEBRAS

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ABSTRACT. This paper introduces *compatible Lie groups*, which are smooth families of Lie groups whose group operations satisfy a certain infinitesimal affineness condition. Answering a problem of Manchon, we construct a tangent space functor from local compatible Lie groups to finite-dimensional compatible Lie algebras. We prove that this functor is an equivalence of categories, thus providing an analogue of Lie's third theorem for compatible Lie algebras.

Certain compatible Lie algebras integrate to global compatible Lie groups; we compute these for all nilpotent compatible Lie algebras up to dimension 4. We discuss a topological obstruction to global integration for compatible Lie algebras in general.

1. INTRODUCTION

A *compatible Lie algebra* is a vector space $\mathfrak{g}$ equipped with two Lie brackets $[-,-], \{-,-\}$ such that every $\mathbb{F}$-linear combination of the two brackets is itself a Lie bracket. During the European Non-Associative Algebra Seminar in 2026 [17], Dominique Manchon posed the question of what differential-geometric objects have compatible Lie algebras as tangent spaces. The purpose of this article is to propose such objects, which we call *compatible Lie groups*, and provide an analogue of the local Lie group–Lie algebra correspondence for compatible Lie algebras.

Loosely speaking, we define a (local) compatible Lie group structure on a pointed manifold (M, e) over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ to be a smooth one-parameter family of (local) Lie groups $(M, e, *_t)_{t\in\mathbb{F}}$ such that the quadratic term of the Taylor expansion of $*_t$ around (e, e) varies affinely in $t \in \mathbb{F}$. This condition guarantees that the corresponding Lie algebras $(T_eM, [[-,-]]_t)$ also vary affinely in the sense that $[[-,-]]_t = [-,-] + t\{-,-\}$ for some fixed Lie brackets $[-,-], \{-,-\}$, which turn out to be canonical (see Section 4.1.1). For details, see Definition 3.1 and the subsequent discussion.

1.1. **Main result.** In its modern formulation (see, for example, [25]), Lie's third theorem states that the tangent space functor $\mathcal{L}ie$ from local Lie groups to finite-dimensional Lie algebras is an equivalence of categories. Thus, the theory of local Lie groups reduces to the theory of Lie algebras, and vice versa. Our main result is an analogue of Lie's third theorem for compatible Lie algebras.

Theorem 1.1 (Theorems 4.1 and 5.1)**.** *Let $\mathbb{F}$ denote the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Then there exists a tangent space functor $\mathcal{L}\colon \mathsf{LCLG}_{\mathbb{F}} \to \mathsf{CLA}_{\mathbb{F}}$ from local compatible Lie groups to finite-dimensional compatible Lie algebras, and $\mathcal{L}$ is an equivalence of categories.*

In particular, every finite-dimensional compatible Lie algebra $\mathfrak{g}$ integrates to a local compatible Lie group $\mathcal{L}^{-1}(\mathfrak{g})$, and $\mathcal{L}^{-1}(\mathfrak{g})$ is the unique such local compatible Lie group up to isomorphism. We define the tangent space functor $\mathcal{L}$ in Section 4.1.1 and construct $\mathcal{L}^{-1}(\mathfrak{g})$ in Section 4.2.2.

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1.2. **Motivations.** Given a category of algebras that satisfy a Jacobi-type identity, it is natural to seek interpretations of those algebras as tangent spaces of geometric objects. Lie's third theorem and its global incarnation, called the Cartan–Lie theorem, solve this problem for Lie algebras.

The corresponding problem for Leibniz algebras is the widely studied "coquecigrue" problem (see, for example, [13, 23]), which Loday [19] posed in 1993. In 2013, Covez [4] proved an analogue of Lie's third theorem for Leibniz algebras using the category of local augmented Lie racks. Another example is the relationship between post-Lie groups (also called Lie skew braces) and post-Lie algebras; the problem of determining which post-Lie algebras integrate to post-Lie groups was posed in 2024 [2] and remains open (see [5]).

Dominique Manchon posed the corresponding problem for compatible Lie algebras during the European Non-Associative Algebra Seminar in 2026 [17]. As far as the author is aware, the present article is the first published work in this direction. That said, methods from complex algebraic geometry have been applied to study certain classes of compatible Lie algebras [1, 20].

Much of the motivation for studying compatible Lie algebras comes from mathematical physics; for example, see [10, 11, 24]. Compatible Lie algebras are used to study bi-Hamiltonian systems and compatible Poisson brackets [7, 22]. More simply, the Lie brackets $[[-,-]]_t \coloneqq [-,-] + t\{-,-\}$ with $t \in \mathbb{F}$ may be viewed as infinitesimal deformations of $[-,-]$; this perspective has been applied to classify certain compatible Lie algebras [15, 22] and study their cohomology [18].

Likewise, (local) compatible Lie groups (see Definition 3.1) can be viewed as deformations of (local) Lie groups. Although this approach may be reminiscent of Lie group bundles, compatible Lie groups do not require a total space or any local triviality condition (cf. Example 6.5). Compatible Lie groups are also similar in spirit to Leibniz racks and Lie quandles, which are smooth families of racks coming from mathematical physics that generalize Leibniz algebras and Lie algebras; see [8, 9]. However, the fibers of a compatible Lie group are only related topologically and infinitesimally, while the fibers of a Leibniz rack must also be algebraically compatible with each other.

1.3. **Structure of the paper.** In Section 2, we review definitions relating to compatible Lie algebras. In Section 3, we introduce (local) compatible Lie groups.

In Section 4, we show that the tangent space of a (local) compatible Lie group at the identity is functorially a compatible Lie algebra (Theorem 4.1); conversely, every compatible Lie algebra functorially integrates to a local compatible Lie group (Theorem 4.5). In Section 5, we show that these functors form an equivalence of categories (Theorem 5.1), and we discuss several corollaries.

It is natural to seek a global version of the integration in Theorem 1.1, that is, an analogue of the Cartan–Lie theorem for compatible Lie algebras. In Section 6, we describe a class of compatible Lie algebras that integrate to global compatible Lie groups (Proposition 6.1); we compute these for nilpotent compatible Lie algebras up to dimension 4 (Example 6.4) using the classification in [15]. We discuss an obstacle to global integration for compatible Lie algebras in general (Example 6.5).

1.4. **Notation and conventions.** In this paper, all vector spaces $\mathfrak{g}$ are assumed to be finite-dimensional over a ground field $\mathbb{F}$. Given $t \in \mathbb{F}$ and two Lie brackets $[-,-]$ and $\{-,-\}$, we define

$$[[-,-]]_t \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \qquad [[x,y]]_t \coloneqq [x,y] + t\{x,y\}.$$

Every compatible Lie algebra $\mathfrak{g}$ has its compatible Lie brackets denoted by $[-,-]$ and $\{-,-\}$.

In Sections 3–6, $\mathbb{F}$ denotes either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. If $\mathbb{F} = \mathbb{C}$, then the words "smooth" and "diffeomorphism" are taken to mean "holomorphic" and "biholomorphism," respectively. We denote the Heisenberg group by $H_3(\mathbb{F})$ and the Heisenberg Lie algebra by $\mathfrak{h}(1, \mathbb{F})$.

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2. Preliminaries

We review definitions relating to compatible Lie algebras. These definitions can also be found in [15]. In the following, let $\mathfrak{g}$ be a vector space equipped with two Lie brackets $[-,-]$ and $\{-,-\}$.

2.1. Compatible Lie algebras.

We begin with a well-known observation.

Proposition 2.1 ([15, Prop. 2.1])**.** *With $(\mathfrak{g},[-,-],\{-,-\})$ as above, the following are equivalent:*

(1) Every linear combination of $[-,-]$ and $\{-,-\}$ is a Lie bracket on $\mathfrak{g}$.
(2) For each $t\in\mathbb{F}$, the linear combination $[[-,-]]_t$ is a Lie bracket on $\mathfrak{g}$.
(3) The sum $[[-,-]]_1$ is a Lie bracket on $\mathfrak{g}$.
(4) For all $x,y,z\in\mathfrak{g}$, we have

$$\{[x,y],z\}+\{[y,z],x\}+\{[z,x],y\}+[\{x,y\},z]+[\{y,z\},x]+[\{z,x\},y]=0. \tag{2.1}$$

Equation (2.1) is called the *mixed Jacobi identity*.

Definition 2.2. A *compatible Lie algebra* is a triple $(\mathfrak{g},[-,-],\{-,-\})$ satisfying any of the equivalent conditions in Proposition 2.1.

Definition 2.3. Let $(\mathfrak{g},[-,-]_{\mathfrak{g}},\{-,-\}_{\mathfrak{g}})$ and $(\mathfrak{h},[-,-]_{\mathfrak{h}},\{-,-\}_{\mathfrak{h}})$ be compatible Lie algebras. An $\mathbb{F}$-linear map $f\colon\mathfrak{g}\to\mathfrak{h}$ is called a *compatible Lie algebra homomorphism* if f is both a Lie algebra homomorphism $(\mathfrak{g},[-,-]_{\mathfrak{g}})\to(\mathfrak{h},[-,-])_{\mathfrak{h}})$ and a Lie algebra homomorphism $(\mathfrak{g},\{-,-\}_{\mathfrak{g}})\to(\mathfrak{h},\{-,-\})_{\mathfrak{h}})$.

Remark 2.4. In general, the compatible Lie algebras $(\mathfrak{g},[-,-],\{-,-\})$ and $(\mathfrak{g},\{-,-\},[-,-])$ are not isomorphic. That said, the identity map $\mathrm{id}_{\mathfrak{g}}$ is an example of what some authors call a *skew-homomorphism* between the former and the latter.

Evidently, compatible Lie algebras form a category $\mathsf{CLA}_{\mathbb{F}}$. Recall that all compatible Lie algebras in this paper are assumed to be finite-dimensional.

When there is no possibility of confusion, we may refer to a compatible Lie algebra as its underlying vector space $\mathfrak{g}$.

2.2. Examples and additional definitions.

Example 2.5. For a trivial example of a compatible Lie algebra, let $(\mathfrak{g},[-,-])$ be a Lie algebra, let $\lambda\in k$, and define $\{-,-\}\coloneqq\lambda[-,-]$.

Example 2.6. Let V be any vector space, let $A,B\in\mathrm{End}(V)$, and let $\mathfrak{g}\coloneqq\mathbb{F}\times V$. Then the Lie brackets

$$[(1,0),(0,w)]\coloneqq(0,Aw),\qquad[(0,v),(0,w)]\coloneqq\mathbf{0}$$

and

$$\{(1,0),(0,w)\}\coloneqq(0,Bw),\qquad\{(0,v),(0,w)\}\coloneqq\mathbf{0}$$

make $\mathfrak{g}$ a compatible Lie algebra.

Example 2.7. Let $\mathfrak{g}$ be a three-dimensional vector space with basis vectors e_1, e_2, e_3. Recall that the *Heisenberg Lie algebra* is the Lie algebra $\mathfrak{h}(1,\mathbb{F}) = (\mathfrak{g}, [-,-])$ defined by

$$[e_1, e_2] \coloneqq e_3, \qquad [-, e_3] \equiv \mathbf{0}.$$

Define another Lie bracket $\{-,-\}$ on $\mathfrak{g}$ by

$$\{e_1, e_3\} \coloneqq e_3, \qquad \{-, e_2\} \equiv \mathbf{0}.$$

Then $(\mathfrak{g}, [-,-], \{-,-\})$ is a compatible Lie algebra.

In the following, let $\mathfrak{g}$ be a compatible Lie algebra.

Definition 2.8. Let $\mathfrak{h} \leq \mathfrak{g}$ be a subspace. We say that $\mathfrak{h}$ is a *subalgebra* of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}], \{\mathfrak{h}, \mathfrak{h}\} \subseteq \mathfrak{h}$. In particular, if $[\mathfrak{h}, \mathfrak{g}], \{\mathfrak{h}, \mathfrak{g}\} \subseteq \mathfrak{h}$, then we say that $\mathfrak{h}$ is an *ideal* of $\mathfrak{g}$.

Definition 2.9. We say that $\mathfrak{g}$ is *abelian* if both of its Lie brackets are uniformly zero.

Definition 2.10. Suppose for some $n \geq 0$ that every nested bracket $(x_0, (x_1, (\dots (x_{n-1}, x_n) \dots)$ vanishes, where $x_i \in \mathfrak{g}$ and each $(-,-)$ is either $[-,-]$ or $\{-,-\}$. Then we say that $\mathfrak{g}$ is *(n-step) nilpotent*.

There are several equivalent definitions of nilpotent compatible Lie algebras; see [15, Lem. 3.4].

Remark 2.11 ([15, Ex. 3.7]). If $\mathfrak{g}$ is nilpotent, then the Lie algebras $(\mathfrak{g}, \{-,-\})$ and $(\mathfrak{g}, [[-,-]]_t)$ are nilpotent for all $t \in \mathbb{F}$. However, the converse is not true in general. For example, let $\mathfrak{g}$ be a three-dimensional vector space with basis vectors e_1, e_2, e_3, and define

$$[e_1, e_2] \coloneqq e_3, \quad [-, e_3] \coloneqq 0, \quad \{e_1, e_3\} \coloneqq e_2, \quad \{-, e_2\} \coloneqq 0.$$

Then $(\mathfrak{g}, \{-,-\})$ and $(\mathfrak{g}, [[-,-]]_t)$ are nilpotent for all $t \in \mathbb{F}$ because $(\mathfrak{g}, \{-,-\})$ and $(\mathfrak{g}, [-.-])$ are both isomorphic to the Heisenberg Lie algebra $\mathfrak{h}(1,\mathbb{F})$, but $(\mathfrak{g}, [-,-], \{-,-\})$ is not nilpotent because it is centerless in the sense of [15, Def. 2.8].

3. Compatible Lie groups

Henceforth, let $\mathbb{F}$ denote either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. If $\mathbb{F} = \mathbb{C}$, then by "smooth" we mean "holomorphic."

The following two definitions use the definitions of local Lie groups and morphisms of local Lie groups, which are stated in [21], for example.

Definition 3.1. A *local compatible Lie group* $(M, e, \Omega, \Omega^{-1}, \mu, \iota)$ consists of an n-dimensional pointed manifold germ (M, e), an open subset $\Omega \subseteq \mathbb{F} \times M \times M$ containing $\mathbb{F} \times (e, e)$, an open subset $\Omega^{-1} \subseteq \mathbb{F} \times M$ containing $\mathbb{F} \times \{e\}$, and smooth maps

$$\mu \colon \Omega \to M, \qquad\qquad \iota \colon \Omega^{-1} \to M$$
$$(t, g, h) \mapsto g *_t h, \qquad\qquad (t, g) \mapsto g_t^{-1}$$

satisfying the following axioms:

(1) Fix $t \in \mathbb{F}$, and consider the open sets

$$\Omega_t \coloneqq \{(g, h) \in M \times M \mid (t, g, h) \in \Omega\}, \qquad \Omega_t^{-1} \coloneqq \{g \in M \mid (t, g) \in \Omega^{-1}\}.$$

Then the sextuple $(M, e, \Omega_t, \Omega_t^{-1}, *_t, -_t^{-1})$ is a local Lie group, which we call the *fiber* of $(M, e, \Omega, \Omega^{-1}, \mu, \iota)$ over t.

(2) There exist an open neighborhood $U \subseteq M$ of e, an open neighborhood $V \subseteq \mathbb{F}^n$ of $\mathbf{0}$, and a chart $\varphi\colon U \xrightarrow{\sim} V$ satisfying the following axioms:
   (a) $\varphi(e) = \mathbf{0}$.
   (b) $(\mathrm{d}\varphi)_e = \mathrm{id}$.
   (c) For each $t \in \mathbb{F}$, let $B_t(x, y)$ denote the quadratic term of the Taylor expansion of the multiplication

$$m_t\colon (\varphi \times \varphi)(\Omega_t \cap (U \times U)) \to V, \qquad (x, y) \mapsto \varphi(\varphi^{-1}(x) *_t \varphi^{-1}(y))$$

   around $(\mathbf{0}, \mathbf{0})$. Then there exist $\mathbb{F}$-bilinear maps $B_0, B\colon \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$ such that $B_t = B_0 + tB$ for all $t \in \mathbb{F}$. That is,

$$m_t(x, y) = x + y + B_0(x, y) + tB(x, y) + O(\|(x, y)\|^3)$$

   for all $t \in \mathbb{F}$ as $(x, y) \to (\mathbf{0}, \mathbf{0})$ in $\mathbb{F}^n \times \mathbb{F}^n$.

For the sake of readability, we will denote local compatible Lie groups $(M, e, \Omega, \Omega^{-1}, \mu, \iota)$ by (M, e, μ). Similarly, for each $t \in \mathbb{F}$, we will denote the fiber of (M, e, μ) over t by $(M, e, *_t)$.

If $\Omega_t = M \times M$ and $\Omega_t^{-1} = M$ for all $t \in \mathbb{F}$, then we call (M, e, μ) a *(global) compatible Lie group*. In this case, we usually denote the fibers by G_t to help distinguish the local and global cases.

Definition 3.2. A *morphism* of (local) compatible Lie groups $(M, e_M, \mu) \to (N, e_N, \nu)$ is a collection $(\varphi_t)_{t\in\mathbb{F}}$ of morphisms of (local) Lie groups $\varphi_t\colon (M, e_M, *_t) \to (N, e_N, *_t)$ such that $(\mathrm{d}\varphi_s)_e = (\mathrm{d}\varphi_t)_e$ for all $s, t \in \mathbb{F}$.

Note that the collection $(\mathrm{id}_M)_{t\in\mathbb{F}}$ is a morphism of local compatible Lie groups, and morphisms can be composed in the obvious way. Thus, local compatible Lie groups form a category denoted by $\mathsf{LCLG}_{\mathbb{F}}$. The same goes for morphisms of global compatible Lie groups, of course.

3.1. **Intuition and examples.** In Definition 3.2, the purpose of the condition $(\mathrm{d}\varphi_s)_e = (\mathrm{d}\varphi_t)_e$ is to ensure the existence of a well-defined tangent space functor $\mathcal{L}$ (see Proposition 4.3). Although this condition is quite strong, it is also the only relationship we require between the different members φ_t of a morphism $(\varphi_t)_{t\in\mathbb{F}}$. Theorem 1.1 states that this notion of morphisms has an optimal level of flexibility for obtaining a local compatible Lie group–compatible Lie algebra correspondence.

More importantly, axiom (2c) in Definition 3.1 states that the quadratic term $B_t = B_0 + tB$ of the Taylor expansion of m_t around $(\mathbf{0}, \mathbf{0})$ varies affinely in $t \in \mathbb{F}$. On the other hand, for each $t \in \mathbb{F}$, recall from Lie group theory (see, for example, [14, Exer. 3.2]) that the multiplication in coordinates m_t induces a Lie bracket $[[-,-]]_t$ on $\mathcal{L}ie(M, e, *_t) = T_eM$ defined by

$$[[x, y]]_t \coloneqq B_t(x, y) - B_t(y, x), \tag{3.1}$$

where B_t is the aforementioned quadratic term.

Loosely speaking, this means that the tangent space T_eM of a compatible Lie group (M, e, μ) should have a family of Lie brackets $([[-,-]]_t)_{t\in\mathbb{F}}$ that vary affinely in $t \in \mathbb{F}$ in the sense that $[[-,-]]_t = [-,-] + t\{-,-\}$ for some fixed Lie brackets $[-,-], \{-,-\}$. If this holds, then the latter two Lie brackets make T_eM into a compatible Lie algebra. Before we make this precise and prove that it holds in general (see Theorem 4.1), let us consider three examples to build intuition.

Example 3.3. A trivial example of a local compatible Lie group is obtained by taking any (local) Lie group (G, e) with Lie algebra $(\mathfrak{g}, [-,-])$ and defining $g *_t h \coloneqq gh$ for all $t \in \mathbb{F}$. We call (G, e, μ) a *constant compatible Lie group*. The corresponding compatible Lie brackets are $[-,-]$ and 0.

Example 3.4. Let V be a (finite-dimensional) vector space over $\mathbb{F}$, let $A, B \in \mathrm{End}(V)$, and let $M \coloneqq \mathbb{F} \times V$. For each $t \in \mathbb{F}$, consider the external semidirect product $\mathbb{F} \ltimes_t V$ determined by the one-parameter subgroup

$$\mathbb{F} \to \mathrm{GL}(V), \qquad x \mapsto \exp(x(A + tB)).$$

Equivalently, the group operation of $\mathbb{F} \ltimes_t V$ is

$$(x, v) *_t (y, w) \coloneqq (x + y, v + \exp(x(A + tB))w).$$

This defines a compatible Lie group $(M, \mathbf{0}, \mu)$. Indeed, for each $t \in \mathbb{F}$, the quadratic term of the Taylor expansion of m_t around $(\mathbf{0}, \mathbf{0})$ is

$$B_t((x, v), (y, w)) = (0, x(A + tB)w),$$

which has the desired form.

For each $t \in \mathbb{F}$, the Lie algebra of the Lie group $\mathbb{F} \ltimes_t V$ is

$$(\mathbb{F} \times V, [[-, -]]_t), \quad [[(1, 0), (0, w)]]_t \coloneqq (0, (A + tB)w), \quad [[(0, v), (0, w)]]_t \coloneqq \mathbf{0}.$$

Hence, the tangent space of the compatible Lie group $(M, \mathbf{0}, \mu)$ is the compatible Lie algebra from Example 2.6.

Example 3.5. Define a family of Lie groups $(\mathbb{F}^2 \ltimes_t \mathbb{F})_{t\in\mathbb{F}}$ as follows. Let $\mathbb{F}^2 \ltimes_0 \mathbb{F} \coloneqq H_3(\mathbb{F})$ denote the Heisenberg group. For all $t \in \mathbb{F}^\times$, consider the external semidirect product $\mathbb{F}^2 \ltimes_t \mathbb{F}$ defined by

$$\mathbf{x} *_t \mathbf{y} \coloneqq \left(x_1 + y_1, x_2 + y_2, x_3 + \frac{1}{t}(\exp(tx_1) - 1)y_2 + \exp(tx_1)y_3 \right).$$

This defines a compatible Lie group $(\mathbb{F}^3, \mathbf{0}, \mu)$. Indeed, for each $t \in \mathbb{F}$, the quadratic term of the Taylor expansion of m_t around $(\mathbf{0}, \mathbf{0})$ is

$$B_t(\mathbf{x}, \mathbf{y}) = (0, 0, x_1y_2 + tx_1y_3),$$

which has the desired form.

For each $t \in \mathbb{F}$, the Lie algebra of the Lie group $\mathbb{F}^2 \ltimes_t \mathbb{F}$ is

$$(\mathbb{F}^3, [[-, -]]_t), \quad [[e_1, e_2]]_t \coloneqq e_3, \quad [[e_1, e_3]]_t \coloneqq te_3, \quad [[e_2, e_3]]_t \coloneqq 0,$$

where e_1, e_2, e_3 denote the standard basis vectors of $\mathbb{F}^3$. Hence, the tangent space of the compatible Lie group $(\mathbb{F}^3, \mathbf{0}, \mu)$ is the compatible Lie algebra from Example 2.7.

3.2. **Canonicity of Lie brackets.** We show that axiom (2c) in Definition 3.1 is independent of the choice of coordinates. More precisely, if a chart φ exists and satisfies the axioms in Definition 3.1, then in fact all charts ψ that satisfy $\psi(e) = \mathbf{0}$ and $(\mathrm{d}\psi)_e = \mathrm{id}$ must also satisfy axiom (2c).

Proposition 3.6. *Let (M, e, μ) be a compatible Lie group, and let $\varphi, \psi\colon U \xrightarrow{\sim} V$ be any two charts satisfying the axioms in Definition 3.1. Let B_t and B'_t be the quadratic terms of the local Lie group multiplication in the coordinates afforded by φ and ψ, respectively. Then there exists a symmetric $\mathbb{F}$-bilinear map $C\colon V \times V \to V$ such that C is independent of $t \in \mathbb{F}$ and $B'_t = B_t + C$ for all $t \in \mathbb{F}$.*

Proof. Define $\chi \coloneqq \psi \circ \varphi^{-1}$. Then $\chi(\mathbf{0}) = \mathbf{0}$ and $(\mathrm{d}\chi)_{\mathbf{0}} = \mathrm{id}$. It follows from Taylor's theorem that

$$\chi(x) = x + Q(x) + O(\|x\|^3), \qquad \chi^{-1}(x) = x - Q(x) + O(\|x\|^3)$$

as $x \to \mathbf{0}$, where $Q\colon V \to V$ is the homogeneous quadratic term of χ. Define

$$C\colon V \times V \to V, \qquad (x, y) \mapsto Q(x + y) - Q(x) - Q(y).$$

Since Q is quadratic, C is symmetric, bilinear, and independent of $t \in \mathbb{F}$.

We claim that $B'_t = B_t + C$ for all $t \in \mathbb{F}$. Indeed, let m_t and m'_t be the local Lie group multiplication maps in the coordinates afforded by φ and ψ, respectively. Then

$$\begin{aligned} m'_t(x,y) &= \psi(\psi^{-1}(x) *_t \psi^{-1}(y)) \\ &= \chi(m_t(\chi^{-1}(x), \chi^{-1}(y))) \\ &= \chi(x + y - Q(x) - Q(y) + B_t(x,y)) + O(\|(x,y)\|^3) \\ &= x + y + B_t(x,y) + C(x,y) + O(\|(x,y)\|^3) \end{aligned}$$

as $(x,y) \to (\mathbf{0},\mathbf{0})$ in $(\psi \times \psi)(\Omega_t \cap (U \times U))$. But $m'_t(x,y) = x + y + B'_t(x,y) + O(\|(x,y)\|^3)$ as $(x,y) \to (\mathbf{0},\mathbf{0})$, so comparing quadratic terms finishes the proof. □

In fact, Proposition 3.6 and its proof show the following.

Corollary 3.7. *Let (M, e, μ) be a local compatible Lie group. Then for each $t \in \mathbb{F}$, the Lie algebra $(T_eM, [[-,-]]_t)$ defined in (3.1) is canonical, that is, independent of the choice of chart φ.*

Corollary 3.8. *In the context of Definition 3.1, let $\varphi, \psi \colon U \xrightarrow{\sim} V \subseteq \mathbb{F}^n$ be charts satisfying all axioms except possibly (2c). Then φ satisfies (2c) if and only if ψ satisfies (2c).*

4. Compatible Lie calculus

In this section, we prepare for the proof of Theorem 1.1 in the subsequent section. Namely, we prove that the tangent space of any local compatible Lie group (M, e, μ) at the identity is a compatible Lie algebra $\mathcal{L}(M, e, \mu) \coloneqq (T_eM, [-,-], \{-,-\})$; conversely, every compatible Lie algebra $(\mathfrak{g}, [-,-], \{-,-\})$ integrates to a local compatible Lie group $\mathcal{L}^{-1}(\mathfrak{g}) \coloneqq (\mathfrak{g}, \mathbf{0}, \mu)$.

Let $\mathsf{LCLG}_\mathbb{F}$ be the category of local compatible Lie groups, and let $\mathsf{CLA}_\mathbb{F}$ be the category of finite-dimensional compatible Lie algebras over $\mathbb{F}$. We define the functors $\mathcal{L} \colon \mathsf{LCLG}_\mathbb{F} \to \mathsf{CLA}_\mathbb{F}$ and $\mathcal{L}^{-1} \colon \mathsf{CLA}_\mathbb{F} \to \mathsf{LCLG}_\mathbb{F}$ mentioned above; they turn out to be mutually inverse (see Theorem 5.1).

4.1. Compatible Lie differentiation.

We define the functor $\mathcal{L} \colon \mathsf{LCLG}_\mathbb{F} \to \mathsf{CLA}_\mathbb{F}$. We imitate the construction of the usual functor $\mathcal{L}ie$ from local Lie groups to Lie algebras, which is defined by $\mathcal{L}ie(G) \coloneqq \mathfrak{g}$ and $\mathcal{L}ie(\varphi) \coloneqq (\mathrm{d}\varphi)_e$.

4.1.1. *The functor $\mathcal{L}$.* Given a local compatible Lie group (M, e, μ) and a chart φ satisfying the conditions of Definition 3.1, equip the tangent space T_eM with the $\mathbb{F}$-bilinear maps

$$[x,y] \coloneqq B_0(x,y) - B_0(y,x), \qquad \{x,y\} \coloneqq B(x,y) - B(y,x).$$

Equivalently,

$$[x,y] = [[x,y]]_0, \qquad \{x,y\} = [[x,y]]_1 - [x,y],$$

where $[[x,y]]_0$ and $[[x,y]]_1$ are the Lie brackets defined in (3.1).

By Corollary 3.7, the definitions of $[-,-]$ and $\{-,-\}$ are independent of the choice of chart φ. In other words, the construction of the maps $[-,-], \{-,-\}$ from (M, e, μ) is canonical, so we can attempt to define the functor $\mathcal{L}$ on objects by

$$\mathcal{L}(M, e, \mu) \coloneqq (T_eM, [-,-], \{-,-\}).$$

On the other hand, morphisms $(\varphi_t)_{t \in \mathbb{F}}$ in $\mathsf{LCLG}_\mathbb{F}$ consist of morphisms φ_t of germs of pointed manifolds, so we can attempt to define $\mathcal{L}$ on morphisms by $\mathcal{L}(\varphi) \coloneqq (\mathrm{d}\varphi_t)_e$ for any choice of $t \in \mathbb{F}$; by definition, all choices are the same.

4.1.2. *Functoriality.* To show that $\mathcal{L}$ is a functor, we only have to verify that $\mathcal{L}$ actually lands in $\mathsf{CLA}_{\mathbb{F}}$; functoriality follows from the chain rule. In the following, recall that if $(\mathfrak{g}, \langle -, - \rangle)$ is a nonassociative $\mathbb{F}$-algebra, then the *Jacobiator* of $\langle -, - \rangle$ is defined to be the cyclic sum

$$\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \qquad (x, y, z) \mapsto \langle \langle x, y \rangle, z \rangle + \langle \langle y, z \rangle, x \rangle + \langle \langle z, x \rangle, y \rangle.$$

The Jacobiator vanishes everywhere if and only if $\langle -, - \rangle$ satisfies the Jacobi identity.

Theorem 4.1. *Let (M, e, μ) be a local compatible Lie group. Then $\mathcal{L}(M, e, \mu)$ is a compatible Lie algebra. Moreover, for all $t \in \mathbb{F}$, we have $\mathcal{L}ie(M, e, *_t) = (T_eM, [[-, -]]_t)$ as defined in (3.1).*

Proof. We established the final claim in Section 3.1. In particular, the final claim holds for $t = 0, 1$, so the first claim reduces to the statement that $\{-, -\}$ is a Lie bracket. Bilinearity and skew-symmetry follow because $[[-, -]]_1$ and $[-, -] = [[-, -]]_0$ are Lie brackets.

It remains to show that $\{-, -\}$ satisfies the Jacobi identity. Fix $x, y, z \in T_eM$, and define $J_{x,y,z} \colon \mathbb{F} \to T_eM$ by sending $t \in \mathbb{F}$ to the Jacobiator of $[[-, -]]_t$ at (x, y, z). Since each $[[-, -]]_t$ is a Lie bracket, $J_{x,y,z}$ vanishes. On the other hand, axiom (2c) implies that $J_{x,y,z} \in T_eM \otimes_{\mathbb{F}} \mathbb{F}[t]$ is a quadratic polynomial in t (with coefficients in T_eM).

Since $J_{x,y,z}$ vanishes, its quadratic term also vanishes. However, the reader can directly compute that the coefficient of the quadratic term equals the Jacobiator of $\{-, -\}$ at (x, y, z); to do this, use the identity $[[-, -]]_t = [-, -] + t\{-, -\}$ obtained from (3.1) and axiom (2c). Since $x, y, z \in T_eM$ were arbitrary, it follows that $\{-, -\}$ satisfies the Jacobi identity. ☐

Remark 4.2. In the above proof, the coefficient of the linear term of $J_{x,y,z}$ is the left-hand side of the mixed Jacobi identity (2.1), so it vanishes. The constant term of $J_{x,y,z}$ is the Jacobiator of the Lie bracket $[-, -]$, so it also vanishes.

Proposition 4.3. *Let $(\varphi_t)_{t \in \mathbb{F}} \colon (M, e_M, \mu) \to (N, e_N, \nu)$ be a morphism of local compatible Lie groups. Then, for some (hence every) $t \in \mathbb{F}$, the derivative $\mathcal{L}(\varphi) \coloneqq (\mathrm{d}\varphi_t)_e$ is a compatible Lie algebra homomorphism $\mathcal{L}(M, e_M, \mu) \to \mathcal{L}(N, e_N, \nu)$.*

Proof. For each $t \in \mathbb{F}$, the map φ_t is a morphism of local Lie groups $(M, e_M, *_t) \to (N, e_N, *_t)$, so $\mathcal{L}ie(\varphi_t) = (\mathrm{d}\varphi_t)_e = \mathcal{L}(\varphi)$ is a Lie algebra homomorphism $\mathcal{L}ie(M, e_M, *_t) \to \mathcal{L}ie(N, e_N, *_t)$. In particular, taking $t = 0, 1$ and using the linearity of $\mathcal{L}(\varphi)$ completes the proof. ☐

Together with the chain rule, Theorem 4.1 and Proposition 4.3 complete the proof that $\mathcal{L}$ is a functor.

4.2. **Compatible Lie integration.** We define a functor $\mathcal{L}^{-1} \colon \mathsf{CLA}_{\mathbb{F}} \to \mathsf{LCLG}_{\mathbb{F}}$.

4.2.1. *The functor $\mathcal{L}ie^{-1}$.* First, we review some facts from Lie theory. Given a Lie algebra $(\mathfrak{g}, [-, -])$, recall that the Baker–Campbell–Hausdorff series is the Lie series

$$\mathrm{BCH}_{[-,-]}(X, Y) \coloneqq X + Y + \frac{1}{2}[X, Y] + \sum_{k=3}^{\infty} Z_k(X, Y), \qquad X, Y \in \mathfrak{g}, \tag{4.1}$$

where each Z_k is a homogeneous Lie polynomial of degree k in X and Y with rational coefficients; a formula of Dynkin computes these coefficients (see [3, Sec. 2] for details). That is, each Z_k is a certain $\mathbb{Q}$-linear combination of $k - 1$ nested brackets $[-, -]$ involving X and Y. Thus, the Baker–Campbell–Hausdorff series is universal in the sense that the Lie polynomials Z_k are defined independently of $(\mathfrak{g}, [-, -])$.

There is a well-known equivalence of categories $\mathcal{L}ie^{-1}$ from finite-dimensional Lie algebras to local Lie groups; see, for example, [25, Thm. 2.1]. Namely, given a Lie algebra $(\mathfrak{g}, [-,-])$, view its underlying vector space $\mathfrak{g}$ as a manifold. Then the Baker–Campbell–Hausdorff series converges absolutely in a neighborhood $U \subseteq \mathfrak{g}$ of $\mathbf{0}$. In fact, U makes the triple $\mathcal{L}ie^{-1}(\mathfrak{g}) := (\mathfrak{g}, \mathbf{0}, \mathrm{BCH}_{[-,-]})$ into a local analytic Lie group whose Lie algebra is $\mathfrak{g}$ (see [12, X.3]). The Lie bracket $[-,-]$ is evidently recovered from $\mathcal{L}ie^{-1}(\mathfrak{g})$ from the quadratic term of the local Lie group operation $\mathrm{BCH}_{[-,-]}$ in the usual way (namely by using [14, Exer. 3.2]).

Finally, every Lie algebra homomorphism $f\colon \mathfrak{g} \to \mathfrak{h}$ restricts to a morphism of local Lie groups $\mathcal{L}ie(f)\colon \mathcal{L}ie(\mathfrak{g}) \to \mathcal{L}ie(\mathfrak{h})$ in some neighborhood of $\mathbf{0} \in \mathfrak{g}$; see [12, X.3, Thm. 3.2].

4.2.2. *The functor $\mathcal{L}^{-1}$.* Given a compatible Lie algebra $(\mathfrak{g}, [-,-], \{-,-\})$, consider the local Lie groups $\mathcal{L}ie^{-1}(\mathfrak{g}, [[-,-]]_t)$ with $t \in \mathbb{F}$. We want to make these local Lie groups into the fibers of a local compatible Lie group $\mathcal{L}^{-1}(\mathfrak{g}) := (\mathfrak{g}, \mathbf{0}, \mu)$.

Assuming for now that this is possible, we define $\mathcal{L}^{-1}$ on morphisms $f\colon \mathfrak{g} \to \mathfrak{h}$ as follows. For each $t \in \mathbb{F}$, note that f is a Lie algebra homomorphism from $(\mathfrak{g}, [[-,-]]^{\mathfrak{g}}_t)$ to $(\mathfrak{h}, [[-,-]]^{\mathfrak{h}}_t)$. Define $f_t := \mathcal{L}ie^{-1}(f)$ with respect to these two Lie algebras. Then we define $\mathcal{L}^{-1}(f) := (f_t)_{t\in\mathbb{F}}$. Under our assumption, $\mathcal{L}^{-1}$ is manifestly a functor.

Therefore, defining $\mathcal{L}^{-1}$ reduces to constructing $\mathcal{L}^{-1}(\mathfrak{g})$. Before doing this precisely, we outline our approach. Using an analytic lemma (Lemma 4.4), we construct a suitable open neighborhood $\Omega \subseteq \mathbb{F} \times \mathfrak{g} \times \mathfrak{g}$ of $\mathbb{F} \times \{(\mathbf{0}, \mathbf{0})\}$. Then we define

$$\mu\colon \Omega \to \mathfrak{g}, \qquad (t, X, Y) \mapsto \mathrm{BCH}_{[[-,-]]_t}(X, Y) \tag{4.2}$$

and $\mathcal{L}^{-1}(\mathfrak{g}) := (\mathfrak{g}, \mathbf{0}, \mu)$. In Theorem 4.5 below, we prove that $\mathcal{L}^{-1}(\mathfrak{g})$ is a local compatible Lie group such that $\mathcal{L}(\mathcal{L}^{-1}(\mathfrak{g})) = \mathfrak{g}$.

In the following, given a metric space A, an element $x \in A$, and a positive number $r > 0$, let $B_r(x)$ denote the open ball of radius r around x in A.

Lemma 4.4. *Let $\mathfrak{g}$ be a compatible Lie algebra, and let $K \subset \mathbb{F}$ be any nonempty compact subset. Then there exists an open neighborhood U_K of $\mathbf{0} \in \mathfrak{g}$ such that the function $\mu_K\colon K \times U_K \times U_K \to \mathfrak{g}$ defined by the same formula as in (4.2) converges absolutely and uniformly on its domain.*

Proof. Recall that $\mathfrak{g}$ is assumed to be finite-dimensional. By Ado's theorem, we may assume for some $n \geq 0$ that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{F})$ equipped with a matrix norm $\|\cdot\|_{\mathfrak{g}}$. Let $\|\cdot\|_V$ denote the induced operator norm on $V := \mathrm{Bil}_{\mathbb{F}}(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$. Then the map $\beta\colon \mathbb{F} \to V$ defined by $t \mapsto [[-,-]]_t$ is continuous (even affine), so $\beta(K)$ is compact. In particular, the quantity $M_K := \sup_{t\in K} \|\beta(t)\|_V$ is finite.

If $M_K = 0$, then each Lie algebra $(\mathfrak{g}, [[-,-]]_t)$ with $t \in K$ is abelian, so the claim is trivial. Otherwise, let

$$0 < r_K < r'_K < \frac{\log 2}{2M_K}, \quad U'_K := B_{r'_K}(\mathbf{0}) \subseteq \mathfrak{g}, \quad U_K := B_{r_K}(\mathbf{0}) \subseteq U'_K.$$

Fix $t \in K$, and consider the rescaled Lie bracket $\langle -,- \rangle_t := \frac{1}{M_K}[[-,-]]_t$. Then $(\mathfrak{g}, \langle -,- \rangle, \|\cdot\|_{\mathfrak{g}})$ is a Banach Lie algebra because $\mathfrak{g}$ is finite-dimensional. Therefore, a well-known result (see, for example, [3, Prop. 2.2]) states that the Baker–Campbell–Hausdorff series $\mathrm{BCH}_{\langle -,- \rangle_t}(X, Y)$ converges absolutely whenever $\|X\|_{\mathfrak{g}} + \|Y\|_{\mathfrak{g}} < (\log 2)/2$. Since each Z_k in the Baker–Campbell–Hausdorff formula is a homogeneous Lie polynomial, it follows that $\mathrm{BCH}_{[[-,-]]_t}$ converges absolutely on $U'_K \times U'_K$.

By the Weierstrass M-test, $\mathrm{BCH}_{[[-,-]]_t}$ converges absolutely and uniformly on $U_K \times U_K$. The claim follows because $t \in K$ was arbitrary and U_K is defined independently of t. □

Theorem 4.5. *For every compatible Lie algebra $\mathfrak{g}$, there exists a local compatible Lie group $\mathcal{L}^{-1}(\mathfrak{g})$ whose fiber over each $t \in \mathbb{F}$ is the local Lie group $\mathcal{L}ie^{-1}(\mathfrak{g}, [[-,-]]_t)$. Moreover, $\mathcal{L}(\mathcal{L}^{-1}(\mathfrak{g})) = \mathfrak{g}$.*

Proof. For each $t \in \mathbb{F}$, let $K_t \subset \mathbb{F}$ be a compact subset whose interior $\mathrm{int}(K_t)$ contains t, and let U_{K_t} be the corresponding open neighborhood of $\mathbf{0} \in \mathfrak{g}$ from Lemma 4.4. Let

$$\Omega := \bigcup_{t \in \mathbb{F}} \mathrm{int}(K_t) \times U_{K_t} \times U_{K_t},$$

and define μ as in (4.2). We claim that $\mathcal{L}^{-1}(\mathfrak{g}) := (\mathfrak{g}, \mathbf{0}, \mu)$ is a local compatible Lie group.

We directly verify axioms (1) and (2) in Definition 3.1. For each $t \in \mathbb{F}$, the Baker–Campbell–Hausdorff series $\mathrm{BCH}_{[[-,-]]_t}$ agrees on each overlap $(U_{K_r} \times U_{K_r}) \cap (U_{K_s} \cap U_{K_s})$ with $r, s \in \mathbb{F}$ and $t \in \mathrm{int}(K_r) \cap \mathrm{int}(K_s)$, so the local Lie group $\mathcal{L}ie^{-1}(\mathfrak{g}, [[-,-]]_t) = (\mathfrak{g}, \mathbf{0}, \mathrm{BCH}_{[[-,-]]_t})$ is defined on the open subset Ω_t defined in axiom (1). Since the function $t \mapsto [[-,-]]_t$ is affine, Lemma 4.4 implies that μ is a power series in t, X, and Y that converges absolutely and uniformly on each subset $K_s \times U_{K_s} \times U_{K_s}$ with $s \in \mathbb{F}$. Thus, μ is analytic, and axiom (1) holds.

It remains to verify axiom (2). Choose any vector space isomorphism $\varphi \colon \mathfrak{g} \xrightarrow{\sim} \mathbb{F}^n$. For all $t \in \mathbb{F}$, the quadratic term of the Taylor expansion of the local Lie group operation $*_t = \mathrm{BCH}_{[[-,-]]_t}$ is

$$B_t(X,Y) = \frac{1}{2}[[X,Y]]_t = \frac{1}{2}[X,Y] + \frac{t}{2}\{X,Y\},$$

which has the desired form. Hence, $\mathcal{L}^{-1}(\mathfrak{g})$ is a local Lie group. The reader can verify immediately that $\mathcal{L}(\mathcal{L}^{-1}(\mathfrak{g})) = \mathfrak{g}$. □

Now that we have constructed a local integration functor $\mathcal{L}^{-1}$, the reader may wish to revisit Examples 3.4 and 3.5. For more computations with $\mathcal{L}^{-1}$, see Examples 6.4 and 6.5.

5. Main theorem and corollaries

Having now proved that $\mathcal{L}$ and $\mathcal{L}^{-1}$ are functors, we finally provide an analogue of Lie's third theorem for compatible Lie algebras.

5.1. **Proof of the main theorem.** We use the natural isomorphism $\exp\colon \mathcal{L}ie^{-1} \circ \mathcal{L}ie \Rightarrow \mathbf{1}_{\mathsf{LieGrp}}$, which can also be used to prove Lie's third theorem; see [17, Sec. 1]. The components of exp are the exponential maps $\exp_G$ constructed in, for example, [16, p. 518].

For every local Lie group G, the exponential map $\exp_G\colon \mathcal{L}ie(G) \to G$ is a local analytic diffeomorphism near $\mathbf{0} \in \mathcal{L}ie(G)$; see [12, X.3, Thm. 3.2]. The naturality of exp, namely that $\varphi \circ \exp_G = \exp_H \circ \mathcal{L}ie(\varphi)$ for all morphisms of local Lie groups $\varphi\colon G \to H$, is an crucial result in Lie theory; see [12, X.3, Thm. 3.3].

In particular, this result allows one to fully recover the local Lie group operation of G by transporting the Baker–Campbell–Hausdorff series in $\mathcal{L}ie(G)$ across $\exp_G$; see [17, Sec. 1]. This makes $\exp_G$ into an analytic isomorphism of local Lie groups $\mathcal{L}ie^{-1}(\mathcal{L}ie(G)) \xrightarrow{\sim} G$. It follows that exp is a natural isomorphism $\mathcal{L}ie^{-1} \circ \mathcal{L}ie \Rightarrow \mathbf{1}_{\mathsf{LieGrp}}$.

Theorem 5.1 (Theorem 1.1)**.** *The functors $\mathcal{L}$ and $\mathcal{L}^{-1}$ constitute an equivalence of categories.*

Proof. Evidently, $\mathcal{L}(\mathcal{L}^{-1}(f)) = f$ on morphisms $f\colon \mathfrak{g} \to \mathfrak{h}$. It follows from Theorem 4.5 that $\mathcal{L} \circ \mathcal{L}^{-1} = \mathbf{1}_{\mathsf{CLA}_{\mathbb{F}}}$. It remains to construct a natural isomorphism $\mathrm{Exp}\colon \mathcal{L}^{-1} \circ \mathcal{L} \Rightarrow \mathbf{1}_{\mathsf{LCLG}_{\mathbb{F}}}$.

First, we define the components Exp_M of Exp, where (M, e, μ) is a local Lie group. Write $\mathcal{L}^{-1}(\mathcal{L}(M)) = (T_eM, \mathbf{0}, \nu)$. We have to construct an isomorphism of local compatible Lie groups $\mathrm{Exp}_M\colon (T_eM, \mathbf{0}, \nu) \xrightarrow{\sim} (M, e, \mu)$. For each $t \in \mathbb{F}$, denote the corresponding local Lie group by $M_t \coloneqq (M, e, *_t)$, and consider the exponential map $\exp_{M_t}$ as a morphism of local Lie groups (see the above discussion). The domain of $\exp_{M_t}$ is the local Lie group

$$\mathcal{L}ie^{-1}(\mathcal{L}ie(M_t)) = \mathcal{L}ie^{-1}(T_eM, [[-,-]]_t) = (T_eM, \mathbf{0}, \mathrm{BCH}_{[[-,-]]_t}) = (T_eM, \mathbf{0}, \nu(t, -, -)),$$

where the last equality follows from Theorem 4.5. Therefore, we can define

$$\mathrm{Exp}_M \coloneqq (\exp_{M_t})_{t \in \mathbb{F}}.$$

By Lie group theory (see the above discussion), each $\exp_{M_t}$ is an analytic isomorphism of local Lie groups $\mathcal{L}ie(M_t) \xrightarrow{\sim} M_t$, and $(\mathrm{d}\exp_{M_t})_{\mathbf{0}} = \mathrm{id}_{T_eM}$ for all $t \in \mathbb{F}$. Hence, Exp_M is an isomorphism of local compatible Lie groups.

It remains to show that Exp is a natural transformation. But this follows straightforwardly from the naturality of $\exp\colon \mathcal{L}ie^{-1} \circ \mathcal{L}ie \Rightarrow \mathbf{1}_{\mathsf{LieGrp}}$. □

5.2. **Consequences of Theorem 5.1.** Just as in ordinary Lie theory, we immediately deduce the following.

Corollary 5.2. *Let $\varphi\colon (M, e_M, \mu) \hookrightarrow (N, e_N, \nu)$ be a closed embedding of local compatible Lie groups. Then the induced map $(\mathrm{d}\varphi)_e\colon T_eM \hookrightarrow T_eN$ is an embedding of compatible Lie algebras. In particular, if $(\varphi(M), e_N, *_t) \trianglelefteq (N, e_N, *_t)$ for all $t \in \mathbb{F}$, then $(\mathrm{d}\varphi)_e(T_eM)$ is an ideal of T_eN.*

Recall that a local Lie group is abelian if and only if its Lie algebra is abelian; the reader can readily verify this using the Baker–Campbell–Hausdorff series. This fact and Proposition 2.1 provide a straightforward proof of the following. That said, the equivalence of the first two statements may be surprising; this illustrates the strength of Theorem 1.1.

Corollary 5.3. *Let (M, e, μ) be a local compatible Lie group. The following are equivalent:*

(1) For each $t \in \mathbb{F}$, the local Lie group $(M, e, *_t)$ is abelian.
(2) The local Lie groups $(M, e, *_0)$ and $(M, e, *_1)$ are abelian.
(3) The compatible Lie algebra $\mathcal{L}(M, e, \mu)$ is abelian.

We similarly obtain the following.

Corollary 5.4. *Let (M, e, μ) be a local compatible Lie group, and let $\mathfrak{g} \coloneqq \mathcal{L}(M, e, \mu)$. Then, for all $t \in \mathbb{F}$, the fiber $(M, e, *_t)$ is nilpotent if and only if the Lie algebra $(\mathfrak{g}, [[-,-]]_t)$ is nilpotent. In particular, every fiber of (M, e, μ) is nilpotent if $\mathfrak{g}$ is nilpotent as a compatible Lie algebra.*

Remark 5.5. Remark 2.11 provides a counterexample to the converse of the final statement of Corollary 5.4.

6. Global integration

By replacing the category of local Lie groups in Lie's third theorem with the category of connected, simply connected Lie groups, one obtains the celebrated Cartan–Lie theorem; see, for example, [14, Cor. 3.44]. (Confusingly, some authors refer to the Cartan–Lie theorem as "Lie's third theorem.") Since Theorem 1.1 analogizes Lie's third theorem for compatible Lie algebras, one may seek a similar analogy of the Cartan–Lie theorem. In this section, we introduce a class of compatible Lie algebras that do admit such an analogy—including all nilpotent compatible Lie algebras—and construct a compatible Lie algebra that does not.

6.1. **Sufficient conditions for global integration.** Given a Lie algebra $\mathfrak{g}$, let G denote the connected, simply connected Lie group with $\mathcal{L}ie(G) = \mathfrak{g}$. Recall that the the exponential map $\exp_G\colon \mathfrak{g} \to G$ is a local diffeomorphism near $\mathbf{0} \in \mathfrak{g}$, and $\mathfrak{g}$ is called *exponential* if $\exp_G$ is a global diffeomorphism. For example, all nilpotent Lie algebras are exponential. Several equivalent definitions of exponential Lie algebra are given in [6, Prop. 4.1].

The following class of compatible Lie algebras admits a global integration into the category of global compatible Lie groups.

Proposition 6.1. *Let $\mathfrak{g}$ be a compatible Lie algebra. For all $t \in \mathbb{F}$, suppose that the Lie algebra $\mathfrak{g}_t := (\mathfrak{g}, [[-,-]]_t)$ is exponential, and let $(G_t, \cdot_t)$ be the connected, simply connected global Lie group with $\mathcal{L}ie(G_t) = \mathfrak{g}_t$. If the map*

$$\mathcal{E}\colon \mathbb{F} \times \mathfrak{g} \to \coprod_{t\in\mathbb{F}} G_t, \qquad (t, X) \mapsto \exp_{G_t}(X)$$

is a diffeomorphism, then the local compatible Lie group $\mathcal{L}^{-1}(\mathfrak{g})$ afforded by Theorem 4.5 is isomorphic to a global compatible Lie group of the form $(\mathfrak{g}, \mathbf{0}, \mu)$.

Proof. Since $\mathcal{E}$ is a diffeomorphism, we can define

$$\mu\colon \mathbb{F} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \qquad (t, X, Y) \mapsto X *_t Y := \pi_2(\mathcal{E}^{-1}(\mathcal{E}(t, X) \cdot_t \mathcal{E}(t, Y))),$$

where $\pi_2\colon \mathbb{F} \times \mathfrak{g} \twoheadrightarrow \mathfrak{g}$ is the projection onto the second coordinate. Then μ is smooth, and $(\mathfrak{g}, \mathbf{0}, *_t)$ is a global Lie group for all $t \in \mathbb{F}$. The rest of the proof is identical to the final paragraph of the proof of Theorem 4.5; the isomorphism $(\mathfrak{g}, \mathbf{0}, \mu) \cong \mathcal{L}^{-1}(\mathfrak{g})$ is clear. □

Corollary 6.2. *Let $\mathfrak{g}$ be a compatible Lie algebra. Suppose that there exists an integer $n \geq 0$ such that, for all $t \in \mathbb{F}$, the Lie algebra $\mathfrak{g}_t := (\mathfrak{g}, [[-,-]]_t)$ is n-step nilpotent. Then the local compatible Lie group $\mathcal{L}^{-1}(\mathfrak{g})$ afforded by Theorem 4.5 is isomorphic to a connected, simply connected global compatible Lie group $(\mathfrak{g}, \mathbf{0}, \mu)$ whose fibers are nilpotent affine algebraic Lie groups. In particular, this result holds if $\mathfrak{g}$ is nilpotent.*

Proof. For all $t \in \mathbb{F}$, the Baker–Campbell–Hausdorff series $\mathrm{BCH}_{[[-,-]]_t}(X, Y)$ is a polynomial in X and Y. Thus, $(G_t, *_t) := (\mathfrak{g}, \mathbf{0}, \mathrm{BCH}_{[[-,-]]_t})$ is a connected, simply connected global affine algebraic Lie group, and $\mathcal{L}ie(G_t) = \mathfrak{g}_t$. Each G_t is nilpotent, and we can identify $\coprod_{t\in\mathbb{F}} G_t = \mathbb{F} \times \mathfrak{g}$ as sets.

It suffices to show that the map $\mathcal{E}$ defined in Proposition 6.1 is the identity map. This reduces to showing that $\exp_{G_t} = \mathrm{id}_{\mathfrak{g}}$ for all $t \in \mathbb{F}$. To that end, fix $X \in G_t$, and define $\gamma\colon \mathbb{F} \to G_t$ by $u \mapsto uX$. Since $[[-,-]]_t$ is alternating, the formula for the Baker–Campbell–Hausdorff series shows that

$$\gamma(u) *_t \gamma(v) = \mathrm{BCH}_{[[-,-]]_t}(uX, vX) = (u + v)X = \gamma(u + v)$$

for all $u, v \in \mathbb{F}$, so γ is a one-parameter subgroup. But $\gamma(1) = X$, so [16, p. 518] states that $\exp_{G_t}(X) = \gamma(1) = X$. Since $X \in G_t$ and $t \in \mathbb{F}$ were arbitrary, the proof is complete. □

Remark 6.3. Remark 2.11 provides an example of a non-nilpotent compatible Lie algebra $\mathfrak{g}$ to which Corollary 6.2 nevertheless applies.

Name of $\mathfrak{g}$ in [15]	Nonzero Lie brackets (see [15])	$\alpha(t)$	Comment
$\mathsf{NCL}_{3,1}$	None	0	$\mathcal{L}ie^{-1}(\mathfrak{g}_t) \cong \mathbb{F}^3$ for all $t \in \mathbb{F}$
$\mathsf{NCL}_{3,2}$	$[e_1, e_2] = e_3$	1	$\mathcal{L}ie^{-1}(\mathfrak{g}_t) \cong H_3(\mathbb{F})$ for all $t \in \mathbb{F}$
$\mathsf{NCL}_{3,3}$	$\{e_1, e_2\} = e_3$	t	$\mathcal{L}ie^{-1}(\mathfrak{g}_t) \cong H_3(\mathbb{F})$ for all $t \in \mathbb{F}^\times$
$\mathsf{NCL}^\sigma_{3,4}$, $\sigma \in \mathbb{F}^\times$	$[e_1, e_2] = e_3$, $\{e_1, e_2\} = \alpha e_3$	$1 + \sigma t$	$\mathcal{L}ie^{-1}(\mathfrak{g}_t) \cong H_3(\mathbb{F})$ if $t \neq -1/\sigma$

TABLE 6.1. Descriptions of the fibers $\mathcal{L}ie^{-1}(\mathfrak{g}_t)$ of the connected, simply connected global compatible Lie groups $\mathcal{L}^{-1}(\mathfrak{g})$ that integrate three-dimensional nilpotent compatible Lie algebras $\mathfrak{g}$. See Example 6.4 for details.

Example 6.4. In 2025, Ladra–Leite da Cunha–Lopes [15, Sec. 4.5] classified nilpotent compatible Lie algebras $\mathfrak{g}$ up to dimension 4; we use their results to compute the compatible Lie groups $\mathcal{L}^{-1}(\mathfrak{g})$ corresponding under Corollary 6.2 to each nilpotent $\mathfrak{g}$ with $\dim \mathfrak{g} \leq 4$. If $\dim \mathfrak{g} \leq 2$, then $\mathfrak{g}$ is abelian, so $\mathcal{L}^{-1}(\mathfrak{g})$ is the constant compatible Lie group corresponding to $\mathbb{F}^{\dim \mathfrak{g}}$ (see Example 3.3).

In Tables 6.1 and 6.2, we give a coordinate presentation of each $\mathcal{L}^{-1}(\mathfrak{g})$ in the cases that $\dim \mathfrak{g} = 3$ and $\dim \mathfrak{g} = 4$, respectively. The notation in both tables is as follows. Identify $\varphi\colon \mathfrak{g} \xrightarrow{\sim} \mathbb{F}^{\dim \mathfrak{g}}$ under a choice of basis $\{e_1, \dots, e_{\dim \mathfrak{g}}\}$. Given $t \in \mathbb{F}$ and $\mathbf{x}, \mathbf{y} \in \mathcal{L}ie^{-1}(\mathfrak{g}, [[-,-]]_t)$, denote

$$A \coloneqq x_1y_2 - x_2y_1, \quad B \coloneqq x_2y_3 - x_3y_2, \quad C \coloneqq x_1y_3 - x_3y_1.$$

If $\dim \mathfrak{g} = 3$, then it turns out that $\mathfrak{g}$ is abelian or two-step nilpotent, and each Lie bracket $[[-,-]]_t$ is defined by $[[-, e_3]]_t \equiv \mathbf{0}$ and $[[e_1, e_2]]_t = \alpha(t)e_3$ for some function $\alpha\colon \mathbb{F} \to \mathbb{F}$. Therefore, the group operation of each fiber $\mathcal{L}ie^{-1}(\mathfrak{g}, [[-,-]]_t)$ of $\mathcal{L}^{-1}(\mathfrak{g})$ in φ-coordinates is

$$m_t(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} + \left(0, 0, \frac{\alpha(t)}{2}A\right),$$

so it suffices to specify α.

Similarly, if $\dim \mathfrak{g} = 4$, then it turns out that $\mathfrak{g}$ is abelian, two-step nilpotent, or three-step nilpotent, and each Lie bracket $[[-,-]]_t$ has the form

$$[[e_1, e_2]]_t = \alpha_1(t)e_3 + \alpha_2(t)e_4, \quad [[e_2, e_3]]_t = \beta(t)e_4, \quad [[e_1, e_3]]_t = \gamma(t)e_4, \quad [[-, e_4]]_t \equiv \mathbf{0}$$

for some functions $\alpha_1, \alpha_2, \beta, \gamma\colon \mathbb{F} \to \mathbb{F}$. Therefore, the Lie bracket in φ-coordinates is

$$[[\mathbf{x}, \mathbf{y}]]_t = (0,\ 0,\ \alpha_1(t)A,\ \alpha_2(t)A + \beta(t)B + \gamma(t)C).$$

On the other hand, the cubic term of the Baker–Campbell–Hausdorff series (4.1) is

$$Z_3(X, Y) = \frac{1}{12}([X, [X, Y]] - [Y, [X, Y]]).$$

Using the fact that e_4 is central in $(\mathfrak{g}, [[-,-]]_t)$, the reader can thus compute the group operation of each fiber $\mathcal{L}ie^{-1}(\mathfrak{g}, [[-,-]]_t)$ in φ-coordinates to be

$$m_t(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} + \Bigg(0,\ 0,\ \frac{\alpha_1(t)}{2}A,\ \frac{1}{2}(\alpha_2(t)A + \beta(t)B + \gamma(t)C) + \frac{\alpha_1(t)}{12}A\Big(\beta(t)(x_2 - y_2) + \gamma(t)(x_1 - y_1)\Big)\Bigg).$$

Name of $\mathfrak{g}$ in [15]	Nonzero Lie brackets (see [15])	$\alpha_1(t)$	$\alpha_2(t)$	$\beta(t)$	$\gamma(t)$
$\mathsf{NCL}_{4,1}$	None	0	0	0	0
$\mathsf{NCL}_{4,2}$	$[e_1, e_2] = e_3$	1	0	0	0
$\mathsf{NCL}_{4,3}$	$\{e_1, e_2\} = e_3$	t	0	0	0
$\mathsf{NCL}^{\sigma}_{4,4}$, $\sigma \in \mathbb{F}^\times$	$[e_1, e_2] = e_3$, $\{e_1, e_2\} = \sigma e_3$	$1 + \sigma t$	0	0	0
$\mathsf{NCL}_{4,5}$	$[e_2, e_3] = e_4$, $\{e_1, e_3\} = e_4$	0	0	1	t
$\mathsf{NCL}_{4,6}$	$[e_1, e_2] = e_3$, $[e_2, e_3] = e_4$	1	0	1	0
$\mathsf{NCL}_{4,7}$	$[e_1, e_2] = e_3$, $\{e_2, e_3\} = e_4$	1	0	t	0
$\mathsf{NCL}^{\tau}_{4,8}$, $\tau \in \mathbb{F}^\times$	$[e_1, e_2] = e_3$, $[e_2, e_3] = e_4$ $\{e_2, e_3\} = \tau e_4$	1	0	$1 + \tau t$	0
$\mathsf{NCL}^{\tau}_{4,9}$, $\tau \in \mathbb{F}^\times$	$[e_1, e_2] = e_3$, $[e_2, e_3] = e_4$ $\{e_1, e_3\} = \tau e_4$	1	0	1	τt
$\mathsf{NCL}_{4,10}$	$[e_1, e_2] = e_3$, $[e_2, e_3] = e_4$ $\{e_1, e_2\} = e_4$	1	t	1	0
$\mathsf{NCL}_{4,11}$	$\{e_1, e_2\} = e_3$, $\{e_2, e_3\} = e_4$	t	0	t	0
$\mathsf{NCL}_{4,12}$	$[e_2, e_3] = e_4$, $\{e_1, e_2\} = e_3$	t	0	1	0
$\mathsf{NCL}^{\tau}_{4,13}$, $\tau \in \mathbb{F}^\times$	$[e_2, e_3] = \tau e_4$, $\{e_1, e_2\} = e_3$, $\{e_2, e_3\} = e_4$	t	0	$\tau + t$	0
$\mathsf{NCL}^{\tau}_{4,14}$, $\tau \in \mathbb{F}^\times$	$[e_1, e_3] = \tau e_4$, $\{e_1, e_2\} = e_3$, $\{e_2, e_3\} = e_4$	t	0	t	τ
$\mathsf{NCL}_{4,15}$	$[e_1, e_2] = e_4$, $\{e_1, e_2\} = e_3$, $\{e_2, e_3\} = e_4$	t	1	t	0
$\mathsf{NCL}^{\sigma}_{4,16}$, $\sigma \in \mathbb{F}^\times$	$[e_1, e_2] = e_3$, $[e_2, e_3] = e_4$, $\{e_1, e_2\} = \sigma e_3$	$1 + \sigma t$	0	1	0
$\mathsf{NCL}^{\sigma}_{4,17}$, $\sigma \in \mathbb{F}^\times$	$[e_1, e_2] = e_3$, $\{e_1, e_2\} = \sigma e_3$, $\{e_2, e_3\} = e_4$	$1 + \sigma t$	0	t	0
$\mathsf{NCL}^{\sigma,\tau}_{4,18}$, $\sigma, \tau \in \mathbb{F}^\times$	$[e_1, e_2] = e_3$, $[e_2, e_3] = e_4$, $\{e_1, e_2\} = \sigma e_3$, $\{e_2, e_3\} = \tau e_4$	$1 + \sigma t$	0	$1 + \tau t$	0
$\mathsf{NCL}^{\sigma,\tau}_{4,19}$, $\sigma, \tau \in \mathbb{F}^\times$	$[e_1, e_2] = e_3$, $[e_2, e_3] = e_4$, $\{e_1, e_2\} = \sigma e_3$, $\{e_1, e_3\} = \tau e_4$	$1 + \sigma t$	0	1	τt

TABLE 6.2. Descriptions of the fibers $\mathcal{L}ie^{-1}(\mathfrak{g}_t)$ of the connected, simply connected global compatible Lie groups $\mathcal{L}^{-1}(\mathfrak{g})$ that integrate four-dimensional nilpotent compatible Lie algebras $\mathfrak{g}$. See Example 6.4 for details.

6.2. **Discussion.** The Cartan–Lie theorem states that every finite-dimensional Lie algebra integrates to a connected, simply connected Lie group G, and G is unique up to isomorphism. Although it is tempting to try adapting this statement to compatible Lie algebras $(\mathfrak{g}, [-,-], \{-,-\})$ in a manner similar to Theorem 1.1 or Proposition 6.1, one obstruction is that the connected, simply

Parameter $t \in \mathbb{R}$	Lie algebra $\mathfrak{g}_t := (\mathbb{R}^3, [[-,-]]_t)$	Local Lie group $\mathcal{L}ie^{-1}(\mathfrak{g}_t)$
$\lvert t\rvert < 1$	$\mathfrak{so}(3)$	$\mathrm{SU}(2)$
$\lvert t\rvert > 1$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathrm{SL}_2(\mathbb{R})$
$t = -1$	Euclidean Lie algebra $\mathfrak{e}(2)$	Euclidean group $E(2)$
$t = 1$	$\mathfrak{h}(1,\mathbb{R})$	$H_3(\mathbb{R})$

TABLE 6.3. Isomorphism classes of the fibers $\mathcal{L}ie^{-1}(\mathfrak{g}_t)$ of the local compatible Lie group $\mathcal{L}^{-1}(\mathbb{R}^3, [-,-], \{-,-\})$ in Example 6.5.

Parameter $t \in \mathbb{C}$	Lie algebra $\mathfrak{g}_t := (\mathbb{C}^3, [[-,-]]_t)$	Local Lie group $\mathcal{L}ie^{-1}(\mathfrak{g}_t)$
$t \neq \pm 1$	$\mathfrak{sl}(2,\mathbb{C})$	$\mathrm{SL}_2(\mathbb{C})$
$t = -1$	Euclidean Lie algebra $\mathfrak{e}(2,\mathbb{C})$	Euclidean group $E(2,\mathbb{C})$
$t = 1$	$\mathfrak{h}(1,\mathbb{C})$	$H_3(\mathbb{C})$

TABLE 6.4. Isomorphism classes of the fibers $\mathcal{L}ie^{-1}(\mathfrak{g}_t)$ of the local compatible Lie group $\mathcal{L}^{-1}(\mathbb{C}^3, [-,-], \{-,-\})$ in Example 6.5.

connected Lie groups G_t integrating the Lie algebras $(\mathfrak{g}, [[-,-]]_t)$ need not be diffeomorphic or even homotopy equivalent for different values of $t \in \mathbb{F}$.

Example 6.5. Let e_1, e_2, e_3 be the standard basis vectors of $\mathbb{F}^3$, and consider the Lie brackets

$$[e_1, e_2] := e_3, \quad [e_2, e_3] := e_1, \quad [e_3, e_1] := e_2$$

and

$$\{e_1, e_2\} := e_3, \quad \{e_2, e_3\} := -e_1, \quad \{e_3, e_1\} := -e_2.$$

If $\mathbb{F} = \mathbb{R}$, then $(\mathbb{R}^3, [-,-]) \cong \mathfrak{so}(3)$ and $(\mathbb{R}^3, \{-,-\}) \cong \mathfrak{sl}(2,\mathbb{R})$, both of which complexify to $\mathfrak{sl}(2,\mathbb{C})$. Therefore, if $\mathbb{F} = \mathbb{C}$, then $(\mathbb{C}^3, [-,-])$ and $(\mathbb{C}^3, \{-,-\})$ are both isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. In either case, these two Lie brackets are compatible; in fact, $(\mathbb{F}^3, \frac{1}{2}[[-,-]]_1) = \mathfrak{h}(1,\mathbb{F})$.

Under the Cartan–Lie theorem, the simply connected real Lie groups G_0, G_1 that integrate $(\mathbb{R}^3, [[-,-]]_0)$ and $(\mathbb{R}^3, [[-,-]]_1)$ are $G_0 = \mathrm{SU}(2)$ and $G_1 = H_3(\mathbb{R})$, respectively. These Lie groups are not homotopy equivalent. Similarly, the simply connected complex Lie groups G_0, G_1 that integrate $(\mathbb{C}^3, [[-,-]]_0)$ and $(\mathbb{C}^3, [[-,-]]_1)$ are $G_0 = \mathrm{SL}_2(\mathbb{C})$ and $G_1 = H_3(\mathbb{C})$, respectively. Once again, these Lie groups are not homotopy equivalent.

This obstructs $\mathfrak{g} := (\mathbb{F}^3, [-,-], \{-,-\})$ from integrating to a simply connected global compatible Lie group in a way that analogizes the Cartan–Lie theorem. Of course, this is no longer an issue if we work with local Lie groups instead of global Lie groups. By Theorem 4.5, $\mathfrak{g}$ integrates to a local compatible Lie group $\mathcal{L}^{-1}(\mathfrak{g})$. We describe the isomorphism classes of the Lie algebras $(\mathbb{F}^3, [[-,-]]_t)$ and the corresponding fibers of $\mathcal{L}^{-1}(\mathfrak{g})$ in Table 6.3 for $\mathbb{F} = \mathbb{R}$ and Table 6.4 for $\mathbb{F} = \mathbb{C}$.

Although Theorem 1.1 provides a local integration procedure for compatible Lie algebras, the question of whether a functorial notion of global integration exists for compatible Lie algebras—most desirably in the form of an equivalence of categories in analogy to the Cartan–Lie theorem—remains open. In view of Example 6.5, such a notion may require replacing compatible Lie groups with a more flexible notion of families of global Lie groups that need not be pairwise diffeomorphic.

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